The characteristic polynomial of an algebra and representations

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Suppose that $\mathbf{k}$ is a field and let A be a finite dimensional, associative, unital $\mathbf{k}$-algebra. Often one is interested in studying the finite-dimensional representations of A. Of course, a finite dimensional representation of A is simply a finite dimensional $\mathbf{k}$-vector space M and a $\mathbf{k}$-algebra homomorphism $A \to \mathrm{End}_{\mathbf{k}}(M)$. In this article we will not consider representations of algebras, but rather how to determine if a $\mathbf{k}$-linear map $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ is actually a homomorphism. We restrict our attention to the case where A is a product of copies of $\mathbf{k}$. If $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ is a representation then certainly, if $a \in A$ satisfies $a^m = 1$ then $\phi(a)^m = \mathrm{id}$ as well. Our first Theorem is a remarkable converse to this elementary observation.

Theorem A. *Let $\mathbf{k}$ be a field of characteristic unequal to 2. Let $A = \mathbf{k}^{\times d}$ and B be a finite dimensional $\mathbf{k}$-algebra. Fix $n > 2$ and suppose that $\mathbf{k}$ contains a full set of n^{th} roots of unity. If a linear map $\phi\colon A \to B$ satisfies $\phi(1_A) = 1_B$ and $\phi(a)^n = 1_B$ for each $a \in A$ such that $a^n = 1_A$, then ϕ is an algebra homomorphism.*

Remark 1. In the case $A = \mathrm{Mat}_n(\mathbf{k})$ and $k > 1$, there is a complete classification of those linear mappings $\phi\colon A \to \mathrm{Mat}_m(\mathbf{k})$ such that $\phi(X^k) = \phi(X)^k$ and of those mappings where if $X^k = X$ then $\phi(X)^k = \phi(X)$. See [CLT87, Bv93, ZC06] and the references therein. Such mappings need not be homomorphisms (or antihomomorphisms).

Consider the regular representation $\mu_L : A \to \mathrm{End}_{\mathbf{k}}(A)$ of A on itself by left multiplication. For $a \in A$, let $\chi_a(t)$ and $\overline{\chi}_a(t)$ be the characteristic and minimal polynomials of $\mu_L(a)$, respectively. We note that $\chi_a(a) = \overline{\chi}_a(a) = 0$ in A. Therefore if M is a finite dimensional left A module with structure map $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ then $\chi_a(\phi(a)) = \overline{\chi}_a(\phi(a)) = 0$ in $\mathrm{End}_{\mathbf{k}}(M)$. The notion of assigning a characteristic polynomial to each element of an algebra and considering representations which are compatible with this assignment has appeared in [Pro87]. This idea has been applied to some problems in noncommutative geometry as well [LB03].

Definition 2. Suppose that $\phi : A \to B$ is a $\mathbf{k}$-linear map, where B is a $\mathbf{k}$-algebra. We say that ϕ is a *characteristic morphism* if $\chi_a(\phi(a)) = 0$ for all $a \in A$. We say that ϕ is *minimal-characteristic* if, moreover, $\overline{\chi}_a(\phi(a)) = 0$ for all $a \in A$.

Remark 3. While the notion of characteristic morphism appears to be new, especially in the case where A and B are general $\mathbf{k}$-algebras, several related notions have been studied. Suppose that $A = \mathrm{Mat}_n(\mathbf{k})$ and ϕ is a linear endomorphism of A. If ϕ preserves the determinant then, according to a result of Frobenius [Fro97], $\phi(X) = MXN$ or $\phi(X) = MX^TN$ where $\det(MN) = 1$. Furthermore,

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Marcus and Purves [MP59] extend this by proving that if ϕ preserves any one of the coefficients of the characteristic polynomial (other than the $(n-1)^{st}, (n-2)^{nd}, (n-3)^{rd}$ or 0^{th}) then either $\phi(X) = MXM^{-1}$ or $\phi(X) = MX^TM^{-1}$ (up to multiplying by an appropriate root of unity). Their result implies that a characteristic endomorphism of $A = \mathrm{Mat}_n(\mathbf{k})$ is either an automorphism or anti-automorphism.

It is natural to ask whether or not the notions of characteristic morphism and minimal-characteristic morphism are weaker than the notion of algebra morphism. Let us address minimal-characteristic morphisms first.

Corollary. *Assume that* $\mathbf{k}$ *is a field of characteristic unequal to 2 which has a full set of* d^{th} *roots of unity. Let* B *be a finite dimensional* $\mathbf{k}$*-algebra. Then a minimal-characteristic morphism* $\phi : \mathbf{k}^{\times d} \to B$ *is an algebra morphism.*

Proof. First note that $\overline{\chi}_1(t) = t - 1$, so $\phi(1) = \mathrm{id}$. Furthermore if $a \in A$ satisfies $a^d = 1$ then $\overline{\chi}_a(t)$ divides $t^d - 1$. Therefore, $\phi(a)^d = \mathrm{id}$. Hence, if $d > 2$ then Theorem A implies that ϕ is an algebra morphism. We leave the cases $d = 1, 2$ for the reader. ■

Example 4. In this example, we show that a characteristic morphism need not be an algebra homomorphism. Let $a, b \in \mathbf{k}$ be such that $a + b \neq 0$. Then the map $\phi : \mathbf{k}^{\times 2} \to \mathrm{Mat}_2(\mathbf{k})$ given by

$$\phi(e_1) = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \quad \phi(e_2) = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$$

is a characteristic morphism that is not a representation.

Characteristic morphisms form a category in a natural way. Any linear map $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ endows M with the structure of a $\mathrm{T}(A)$ module, where $\mathrm{T}(A)$ denotes the tensor algebra on A. Let $\chi_A(t) \in \mathrm{Sym}^\bullet(A^\vee)[t]$ be the homogeneous form such that for each $a \in A$, $\chi_a(t) = \mathrm{ev}_a(\chi_A(t))$ under the evaluation morphism

$$\mathrm{ev}_a \colon \mathrm{Sym}^\bullet(A^\vee)[t] \to \mathbf{k}[t].$$

Pappacena [Pap00] associates to such a homogeneous form an algebra

$$C(A) = \frac{\mathrm{T}(A)}{\langle \chi_a(a) : a \in A \rangle},$$

where if $\chi_a(t) = \sum_{i=0}^d c_i(a) t^i$ then

$$\chi_a(a) := \sum_{i=0}^d c_i(a) a^{\otimes i} \in \mathrm{T}(A).$$

Clearly, the action map $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$ of a $\mathrm{T}(A)$-module M is a characteristic morphism if and only if the action of $\mathrm{T}(A)$ factors through $C(A)$. We declare the category of characteristic morphisms to be the category of finite-dimensional $C(A)$-modules. We now define the notion of irreducible characteristic morphism in the usual way. The characteristic morphism constructed in Example 4 is not irreducible, being an extension of two irreducible characteristic morphisms which are themselves algebra morphisms. The following example shows that an irreducible characteristic morphism is not necessarily an algebra morphism.

Example 5. Suppose that $\mathrm{char}\,\mathbf{k} \neq 2$. The linear map $\phi : \mathbf{k}^{\times 3} \to \mathrm{Mat}_3(\mathbf{k})$ defined by

$$e_1 \mapsto \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad e_2 \mapsto \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \quad e_3 \mapsto \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

is an irreducible characteristic morphism, but not an algebra morphism. For example, it can be checked that $\phi(e_1)^2 \neq \phi(e_1)$.

Given a linear map $\phi : A \to \mathrm{End}_{\mathbf{k}}(M)$, let $T_\phi \in A^\vee \otimes \mathrm{End}_{\mathbf{k}}(M)$ be the element that corresponds to ϕ under the isomorphism $\mathrm{Hom}_{\mathbf{k}}(A, \mathrm{End}_{\mathbf{k}}(M)) \cong A^\vee \otimes \mathrm{End}_{\mathbf{k}}(M)$. We view T_ϕ as an element of $\mathrm{Sym}^\bullet(A^\vee) \otimes \mathrm{End}_{\mathbf{k}}(M)$. The equation $\chi_a(\phi(a)) = 0$ for all $a \in A$ holds if and only if $\chi_A(T_\phi) = 0$ in $\mathrm{Sym}^\bullet(A^\vee) \otimes \mathrm{End}_{\mathbf{k}}(M)$, where χ_A is the form defined above. We can just as easily view T_ϕ as an element of $\mathrm{T}(A^\vee) \otimes \mathrm{End}_{\mathbf{k}}(M)$. Moreover, we can lift χ_A from $\mathrm{Sym}^\bullet(A^\vee)[t]$ to $\mathrm{T}(A^\vee) * \mathbf{k}[t]$ by the naïve symmetrization map $\mathrm{Sym}^\bullet(A^\vee)[t] \to \mathrm{T}(A^\vee) * \mathbf{k}[t]$, where $*$ denotes the coproduct of associative $\mathbf{k}$-algebras. By abuse of notation we also denote by χ_A its lift.

Theorem B. *Let A be a finite dimensional, étale $\mathbf{k}$-algebra, B a finite dimensional $\mathbf{k}$-algebra and $\phi\colon A \to B$ a $\mathbf{k}$-linear map. Assume either that $\mathrm{char}\,\mathbf{k} = 0$ or that $\mathrm{char}\,\mathbf{k} > dim(B), dim(A)$. Then the map ϕ is a homomorphism if and only if $\chi_A(T_\phi) = 0$ in $\mathrm{T}(A^\vee) \otimes_{\mathbf{k}} B$.*

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Proofs

We now turn to the proofs of the results in the introduction. The proof of the Theorem A depends on an arithmetic Lemma.

Lemma 6. *Let $\zeta \in \mathbf{k}$ be a primitive n^{th} root of unity. Suppose that $a, b, c, d \in \mathbb{Z}$ satisfy $b, d \neq 0 \mod n$ and*

$$\frac{\zeta^a - 1}{\zeta^b - 1} = \frac{\zeta^c - 1}{\zeta^d - 1}.$$

Then either:

1. $a \equiv b \mod n$ *and* $c \equiv d \mod n$, *or*
2. $a \equiv c \mod n$ *and* $b \equiv d \mod n$.

Proof. After possibly passing to a finite extension we may assume that $\mathbf{k}$ admits an automorphism sending ζ to ζ^{-1}. Thus we have

$$\frac{\zeta^{-a} - 1}{\zeta^{-b} - 1} = \frac{\zeta^{-c} - 1}{\zeta^{-d} - 1},$$

which we rewrite

$$\frac{\zeta^{-a}}{\zeta^{-b}} \cdot \frac{1 - \zeta^a}{1 - \zeta^b} = \frac{\zeta^{-c}}{\zeta^{-d}} \cdot \frac{1 - \zeta^c}{1 - \zeta^d}.$$

Using our assumption we find that $\zeta^{b-a} = \zeta^{d-c}$. Thus $b - a \equiv d - c \mod n$. Let $e = b - a \equiv d - c \mod n$. Then we have

$$\frac{\zeta^{b-e} - 1}{\zeta^b - 1} = \frac{\zeta^{d-e} - 1}{\zeta^d - 1}$$

which implies that

$$\zeta^{b-e} + \zeta^d = \zeta^{d-e} + \zeta^b.$$

Finally we see that

$$\zeta^d - \zeta^b = (\zeta^d - \zeta^b)\zeta^{-e}$$

Therefore either $e \equiv 0 \mod d$ so that (1) holds, or $d \equiv b$ so that (2) holds. ■

Proof of Theorem A. Whether or not ϕ is an algebra homomorphism is unchanged after passing to the algebraic closure of $\mathbf{k}$. So we may assume that $\mathbf{k}$ is algebraically closed. Then $A \cong \mathbf{k}^{\times d}$. Furthermore ϕ is an algebra homomorphism if and only if $\rho \circ \phi \colon A \to \operatorname{End}_{\mathbf{k}}(B)$ is one, where $\rho \colon B \to \operatorname{End}_{\mathbf{k}}(B)$ is the (left) regular representation. So it is sufficient to prove the Theorem in the case where $B = \operatorname{End}_{\mathbf{k}}(M)$ for some finite dimensional $\mathbf{k}$-vector space M. Let $e_1, \ldots, e_d \in A$ be a complete set of orthogonal idempotents. Put $\alpha_i = \phi(e_i)$ and note that by hypothesis $\alpha_1 + \cdots + \alpha_d = \mathrm{id}$. Fix a primitive n^{th} root of unity ξ. Then $x = 1 + (\xi - 1)e_i$ satisfies $x^n = 1$. Therefore $\phi(x)^n = \mathrm{id}$. This implies, since the characteristic of $\mathbf{k}$ does not divide n, that $\phi(x)$ is diagonalizable and each eigenvalue is an n^{th} root of unity. Now, since ϕ is linear,

$$\alpha_i = \frac{\phi(x) - \mathrm{id}}{\xi - 1}$$

and hence α_i is diagonalizable as well. Let λ be an eigenvalue of α_i. Then for some a we have

$$\lambda = \frac{\xi^a - 1}{\xi - 1}.$$

Now for any b, $\phi(1 + (\xi^b - 1)e_i)^n = \mathrm{id}$. So we see that

$$1 + \lambda(\xi^b - 1)$$

must be a root of unity for every b. However, if

$$1 + \lambda(\xi^b - 1) = \xi^c$$

then Lemma 6 implies that either $a \equiv 1 \mod n$, $\lambda = 0$, or $b \equiv 1 \mod n$. Now, b is under our control and since $n \geq 3$ we can choose $b \not\equiv 0, 1 \mod n$, excluding the third case. If $a \equiv 1 \mod n$ then $\lambda = 1$ and otherwise $\lambda = 0$. Thus α_i is semisimple with eigenvalues equal to zero or one. So $\alpha_i^2 = \alpha_i$.

Let $i \neq j$ and consider

$$y_a = \mathrm{id} + (\xi^a - 1)(\alpha_i + \alpha_j)$$

Clearly, $y_a^n = \mathrm{id}$ and thus y_a is semisimple, having eigenvalues that are n^{th} roots of unity. We compute

$$(y_a - \mathrm{id})^2 = (\xi^a - 1)^2(\alpha_i\alpha_j + \alpha_j\alpha_i) + (\xi^a - 1)(y_a - \mathrm{id})$$

and deduce that

$$(\xi^a - 1)^{-2}(y_a - \mathrm{id})(y_a - \xi^a) = (\alpha_i\alpha_j + \alpha_j\alpha_i). \tag{1}$$

Observe that $y_a - \mathrm{id} = \frac{\xi^a - 1}{\xi^b - 1}(y_b - \mathrm{id})$ and therefore, y_a and y_b are simultaneously diagonalizable. Suppose that ξ^c is an eigenvalue of y_b unequal to 1. Then

$$\frac{\xi^a - 1}{\xi^b - 1}(\xi^c - 1) + 1 = \xi^e$$

is an eigenvalue of y_a. Since $n \geq 3$ we can assume that $a \neq b, 0 \mod n$. Then Lemma 6 implies that $e \equiv a \mod n$ and $b \equiv c \mod n$. We see that the only eigenvalues of y_b are 1 and ξ^b.

Because y_b is semisimple, this means that the right side of (1) vanishes. So $\alpha_i\alpha_j = -\alpha_j\alpha_i$ for all i, j. Suppose that $\alpha_i(m) = m$. Then $\alpha_j(\alpha_i(m)) = \alpha_j(m) = -\alpha_i(\alpha_j(m))$. Since -1 is not an eigenvalue of α_i we see that $\alpha_j(m) = 0$. Now let $m \in M$ and write $m = m_0 + m_1$ where $\alpha_i(m_0) = 0$ and $\alpha_i(m_1) = m_1$. Then

$$\alpha_i(\alpha_j(m)) = \alpha_i(\alpha_j(m_0)) = -\alpha_j(\alpha_i(m_0)) = 0.$$

Thus we see that in fact $\alpha_i\alpha_j = 0$. So $\alpha_1, \ldots, \alpha_d$ satisfy the defining relations of $\mathbf{k}^{\times d}$ and ϕ is actually an algebra homomorphism. ■

We now turn to the proof of Theorem B. The key idea is to use the fact that the single equation $\chi_A(T_\phi) = 0$ over the tensor algebra encodes many relations for the matrices defining ϕ. It is convenient to consider $\alpha_i = \phi(e_i)$, where e_i is the standard basis of idempotents in $\mathbf{k}^{\times d}$. Furthermore we write χ_d for the characteristic polynomial of $\mathbf{k}^{\times d}$ viewed as an element of the free associative algebra $\mathbf{k}\langle x_1, \ldots, x_d, t\rangle$ (where $x_1, \ldots, x_d$ is the dual basis to $e_1, \ldots, e_d$).

Lemma 7. *Let $n, d \in \mathbb{N}$ and suppose that $\mathbf{k}$ is either a field of characteristic zero or that $\mathrm{char}(\mathbf{k}) > d, n$. Let $\alpha_1, \ldots, \alpha_d \in \mathrm{Mat}_n(\mathbf{k})$ and put $T = x_1\alpha_1 + \ldots + x_d\alpha_d$. If T satisfies χ_d then*

1. *for some $i = 1, \ldots, d$, α_i has a 1-eigenvector, and*
2. *if $m \in \mathbf{k}^n$ satisfies $\alpha_i m = m$ then $\alpha_j m = 0$ for all $j \neq i$.*

Proof. (1.) Let $S = \mathbf{k}[x_1, \ldots, x_d]$ as an $A = \mathbf{k}\langle x_1, \ldots, x_d\rangle$ module in the obvious way. Then the image of χ_d in $\mathbf{k}[x_1, \ldots, x_d, t]$ is $p(t) = n!(t - x_1)\cdots(t - x_d)$, where now the order of the terms does not matter. Hence T satisfies $(T - x_1)\cdots(T - x_d) = 0$ in $M_n(S)$. So we can view S^n as an $R = \mathbf{k}[x_1, \ldots, x_d, t]/(p(t))$-module M. For each i consider the quotient $S_i := R/(t - x_i)$, which is isomorphic to S under the natural map $S \to S_i$. Define $M_i = M \otimes_R S_i$. Since the map $S \to S_1 \times \cdots \times S_d$ is an isomorphism after inverting $a = \prod_{i \neq j}(x_i - x_j)$ and a is a nonzerodivisor on M, the natural map $M \to M_1 \oplus \cdots \oplus M_d$ is injective. Hence there is some i such that M_i has positive rank. Consider $\bar{M} := M/(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_d)M$ and $\bar{M}_i := M_i/(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_d)M_i$. Now since M_i (is finitely generated and) has positive rank $\bar{M}_i \neq 0$. Observe that since $M = S^d$, the natural map $\mathbf{k}^{\times d} \to \bar{M}$ is an isomorphism. Moreover the action of t on $\bar{M}$ is identified with the action of α_i. Now, $\bar{M}_i = \bar{M}/(t - x_i)\bar{M} = \bar{M}/(\alpha_i - 1)\bar{M} \neq 0$. Hence $\alpha_i - 1$ is not invertible, $\alpha_i - 1$ has nonzero kernel, and α_i has a 1-eigenvector.

(2.) Let us compute $\chi(x_1, \ldots, x_d, T)$. We denote by δ^i_j the Kronecker function. Observe that

$$T - x_i = \sum_{j=1}^{d} x_j(\alpha_j - \delta^i_j).$$

Therefore

$$\begin{aligned}
\chi_d(x_1, \ldots, x_d, T) &= \sum_{\sigma \in S_d} \prod_{i=1}^{d} \left(T - x_{\sigma(i)}\right) \\
&= \sum_{\sigma \in S_d} \prod_{j=1}^{d} \left(\sum_{i=1}^{d} x_i \left(\alpha_i - \delta^i_{\sigma(j)}\right)\right) \\
&= \sum_{1 \leq i_1, \ldots, i_d \leq d} x_{i_1} \cdots x_{i_m} \left(\sum_{\sigma \in S_d} \left(\alpha_{i_1} - \delta^{i_1}_{\sigma(1)}\right) \cdots \left(\alpha_{i_d} - \delta^{i_d}_{\sigma(d)}\right)\right).
\end{aligned}$$

In the second line the term order matters so the product is taken in the natural order $j = 1, 2, \ldots, d$. Now suppose that $\chi_d(x_1, \ldots, x_d, T) = 0$. Then for all $1 \leq i_1, \ldots, i_d \leq d$ we have

$$\sum_{\sigma \in S_d} (\alpha_{i_1} - \delta^{i_1}_{\sigma(1)}) \cdots (\alpha_{i_d} - \delta^{i_d}_{\sigma(d)}) = 0. \tag{2}$$

For each $j \neq i$, we consider the noncommutative monomial $x_i x_j x_i^{d-2}$ and its equation (2),

$$\sum_{\sigma \in S_d} (\alpha_i - \delta^i_{\sigma(1)})(\alpha_j - \delta^j_{\sigma(2)})(\alpha_i - \delta^i_{\sigma(3)}) \cdots (\alpha_i - \delta^i_{\sigma(d)}) = 0. \tag{3}$$

Note that since $\alpha_i(m) = m$, we calculate

$$(\alpha_i - \delta^i_{\sigma(3)}) \cdots (\alpha_i - \delta^i_{\sigma(d)})m = \begin{cases} m & i \notin \{\sigma(3), \dots, \sigma(d)\}, \\ 0 & i \in \{\sigma(3), \dots, \sigma(d)\}. \end{cases}$$

Therefore applying (3) to m and simplifying we get

$$\begin{aligned} \sum_{\sigma \in S_d, \sigma(1)=i} (\alpha_i - 1)(\alpha_j - \delta^j_{\sigma(2)})m + \sum_{\sigma \in S_d, \sigma(2)=i} \alpha_i \alpha_j m &= (d-1)!\big((\alpha_i - 1)(\alpha_j - \delta^j_{\sigma(2)})m + \alpha_i \alpha_j m\big) \\ &= (d-1)!\big((\alpha_i - 1)\alpha_j m + \alpha_i \alpha_j m\big) \\ &= (d-1)!(2\alpha_i - 1)\alpha_j m \\ &= 0, \end{aligned}$$

where passing from the first line to the second we use the fact that $(\alpha_i - 1)\delta^j_{\sigma(2)} m = 0$.

Now, consider the special case of (2) corresponding to x_i^d:

$$\sum_{\sigma \in S_d} (\alpha_i - \delta^i_{\sigma(1)}) \cdots (\alpha_i - \delta^i_{\sigma(d)}) = \sum_{j=1}^{d} \sum_{\sigma \in S_d, \sigma(j)=i} \alpha_i^{j-1}(\alpha_i - 1)\alpha_i^{d-j-1} = d!\alpha_i^{d-1}(\alpha_i - 1) = 0.$$

Since $\alpha_i^{d-1}(\alpha_i - 1) = 0$, zero and one are the only eigenvalues of α_i. In particular, $\frac{1}{2}$ is not an eigenvalue of α_i, so $2\alpha_i - 1$ is invertible. However, $(2\alpha_i - 1)\alpha_j m = 0$ so $\alpha_j m = 0$. ∎

Proof of Theorem B. We first reduce to the case $B \cong \mathrm{End}_{\mathbf{k}}(M)$. Let $\rho\colon B \to \mathrm{End}_{\mathbf{k}}(B)$ be the regular representation. Then ϕ is an algebra homomorphism if and only if $\rho \circ \phi$ is one. Furthermore $T_{\rho\circ\phi}$ is the image of T_ϕ under the injective map $\mathrm{id} \otimes \rho$:

$$\mathrm{T}(A^\vee) \otimes B \to \mathrm{T}(A^\vee) \otimes_{\mathbf{k}} \mathrm{End}_{\mathbf{k}}(B).$$

Therefore $\chi_A(T_\phi) = 0$ if and only if $\chi_A(T_{\rho\circ\phi}) = 0$. Next, we reduce to the case where $A \cong \mathbf{k}^{\times d}$. Note that whether or not $\chi_A(T_\phi) = 0$ in $T(A^\vee) \otimes B$ and whether or not ϕ is an algebra homomorphism are stable under passage to the algebraic closure of $\mathbf{k}$. Since $A \otimes_{\mathbf{k}} \overline{\mathbf{k}} \cong \overline{\mathbf{k}}^{\times d}$, we may assume that $\mathbf{k}$ is algebraically closed and that $A \cong \mathbf{k}^{\times d}$.

($\Leftarrow$) We proceed by induction on $\dim(M)$ and fix an identification $M \cong \mathbf{k}^n$. Suppose $n = 1$. Then by Lemma 7, there is some i and some $m \in \mathbf{k}^1$ such that $\alpha_i(m) = m$. Moreover, $\alpha_j m = 0$ for all $j \neq 0$. Since m spans $\mathbf{k}^1$, the α_i satisfy the neccessary relations for ϕ to factor through an algebra morphism.

Now given $\alpha_1, \dots, \alpha_d \in \mathrm{Mat}_n(\mathbf{k})$, Lemma 7 implies that we can find an element $m \in \mathbf{k}^n$ such that $\mathbf{k}m \subset \mathbf{k}^n$ is stable under the action of $\alpha_1, \dots, \alpha_d$. Let $\mathrm{Mat}_n(\mathbf{k}, m) \subset \mathrm{Mat}_n(\mathbf{k})$ be the algebra of operators that preserve $\mathbf{k}m$. Then there is a surjective algebra homomorphism $\mathrm{Mat}_n(\mathbf{k}, m) \twoheadrightarrow \mathrm{Mat}_{n-1}(\mathbf{k})$. Since $\alpha_1, \dots, \alpha_d \in \mathrm{Mat}_n(\mathbf{k}, m)$ we find that $T \in \mathrm{Mat}_n(\mathbf{k}, m) \otimes_{\mathbf{k}} \mathbf{k}\langle x_1, \dots, x_d\rangle$. So if $\alpha'_1, \dots, \alpha'_d \in \mathrm{Mat}_{n-1}(\mathbf{k})$ are the images of $\alpha_1, \dots, \alpha_d$ then $T' = x_1\alpha'_1 + \ldots + x_d\alpha'_d$ satisfies χ_d. By induction we see that $(\alpha'_i)^2 = \alpha'_i$ and $\alpha'_i\alpha'_j = 0$ for $i \neq j$. In particular, there is a codimension 1 subspace of $\mathbf{k}^{n-1}$ preserved by $\alpha'_1, \dots, \alpha'_d$. Its inverse image in $\mathbf{k}^n$ (we identify $\mathbf{k}^{n-1}$ with $\mathbf{k}^n/\mathbf{k}m$) is then a codimension one subspace $V' \subset \mathbf{k}^{\times d}$ which is invariant under $\alpha_1, \ldots, \alpha_d$. Again by induction, $\alpha_i^2 - \alpha_i$ and $\alpha_i \alpha_j (i \neq j)$ annihilate V'. There is some i such that α_i acts by the identity on $\mathbf{k}^n / V'$. Since $\alpha_i^{d-1}(\alpha_i - 1) = 0$, the geometric multiplicity of 1 as an eigenvalue of α_i is equal to its algebraic multiplicity. So there is a 1-eigenvector $m \in \mathbf{k}^n$ whose image in $\mathbf{k}^n/V'$ is nonzero. Again Lemma 7 implies that $\alpha_j m = 0$ for $j \neq i$. Hence the relations $\alpha_i^2 - \alpha_i$ and $\alpha_i \alpha_j$ annihilate a basis for $\mathbf{k}^n$ and hence annihilate $\mathbf{k}^n$.

($\Rightarrow$) Suppose that ϕ is an algebra map. Then we have $\alpha_i^2 = \alpha_i$ for all i and $\alpha_j \alpha_i = 0$ if $i \neq j$. Decompose $\mathbf{k}^n = V_1 \oplus \ldots \oplus V_d$ where $V_i = \alpha_i(\mathbf{k}^n)$. Then T preserves $V_i \otimes \mathbf{k}\langle x_1, \ldots, x_d \rangle$ for each i. So we can view T as an element of $\prod_{i=1}^d \operatorname{End}_{\mathbf{k}}(V_i) \otimes \mathbf{k}\langle x_1, \ldots, x_d \rangle \subset \operatorname{Mat}_n(\mathbf{k}\langle x_1, \ldots, x_d \rangle)$. Since $(T - x_i)$ vanishes identically on $V_i \otimes \mathbf{k}\langle x_1, \ldots, x_d \rangle$ we see that for each $\sigma \in S_d$ and each i the image of $(T - x_{\sigma(1)}) \cdots (T - x_{\sigma(d)})$ vanishes in $\operatorname{End}_{\mathbf{k}}(V_i) \otimes \mathbf{k}\langle x_1, \ldots, x_d \rangle$ and hence in $M_n(\mathbf{k}\langle x_1, \ldots, x_d \rangle)$. Since all of the terms of $\chi_d(T)$ vanish in $M_n(\mathbf{k}\langle x_1, \ldots, x_d \rangle)$, so does $\chi_d(T)$. ∎

Questions

There are many natural questions that surround the notion of characteristic morphism. We point out a few of them.

Question 1. *What are the irreducible characteristic morphisms for $A = \mathbf{k}^{\times d}$? Are there infinitely many for $d \geq 3$?*

Question 2. *In Theorem B we can replace A with a central simple algebra. If $\phi \colon A \to B$ satisfies $\chi_A(T_\phi) = 0$ in $\mathrm{T}(A^\vee) \otimes B$, must ϕ be an automorphism or anti-automorphism?*

Let V be a finite dimensional vector space and $F(t) \in \operatorname{Sym}^\bullet(V^\vee)[t]$ be monic and homogeneous. Given $v \in V$ we can consider the image $F_v(t)$ of $F(t)$ under the homomorphism $\operatorname{Sym}^\bullet(V^\vee)[t] \to \mathbf{k}[t]$ induced by $v : V^\vee \to \mathbf{k}$. The main theorem of [CK15] implies that there always exists a linear map $\phi : V \to \operatorname{Mat}_r(\mathbf{k})$ for some r such that $F_v(\phi(v)) = 0$ for all $v \in V$. There is a natural non-commutative generalization of this problem.

Question 3. *For which monic, homogenous elements $F(t)$ of $\mathrm{T}(V^\vee) * \mathbf{k}[t]$, does there exist an element $\phi^\vee \in V^\vee \otimes \operatorname{Mat}_r(V)$ for some r such that $F(\phi^\vee) = 0$ in $\mathrm{T}(V^\vee) \otimes \operatorname{Mat}_r(\mathbf{k})$?*

If $F(t)$ is the symmetrization of the characteristic polynomial of an algebra structure on V then we have an affirmative answer. However, if $F(t) = t^2 - u \otimes v$ where u, v are linearly independent, then there is no such element.

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