Vector bundles whose restriction to a linear section is Ulrich

Rajesh S. Kulkarni [*], Yusuf Mustopa [†], and Ian Shipman [‡]

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Abstract

An Ulrich sheaf on an $n-$dimensional projective variety $X \subseteq \mathbb{P}^N$ is an initialized ACM sheaf which has the maximum possible number of global sections. Using a construction based on the representation theory of Roby-Clifford algebras, we prove that every normal ACM variety admits a reflexive sheaf whose restriction to a general 1-dimensional linear section is Ulrich; we call such sheaves $\delta-$Ulrich. In the case $n = 2$, where $\delta-$Ulrich sheaves satisfy the property that their direct image under a general finite linear projection to $\mathbb{P}^2$ is a semistable instanton bundle on $\mathbb{P}^2$, we show that some high Veronese embedding of X admits a $\delta-$Ulrich sheaf with a global section.

Introduction

The structure theory of ACM sheaves on a subvariety $X \subseteq \mathbb{P}^N$ is an important and actively studied area of algebraic geometry. Ulrich sheaves are the "nicest possible" ACM sheaves on X, since their associated Cohen-Macaulay module has the maximum possible number of generators, they are closed under extensions (they form an abelian subcategory of $\mathrm{Coh}(X)$), and their Hilbert series is completely determined by their rank and $\deg(X)$. Moreover, they are all Gieseker-semistable.

Ulrich sheaves on X admit a clean geometric characterization; they are precisely the sheaves whose direct image under a general finite linear projection $\pi : X \to \mathbb{P}^n$ is a trivial vector bundle. In particular, a sheaf on $\mathbb{P}^n$ is Ulrich with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$ if and only if it is a trivial vector bundle. Varieties known to admit Ulrich sheaves include curves and Veronese varieties [ESW03, Han99], complete intersections [BHU91], generic linear determinantal varieties [BHU87], Segre varieties [CMRPL12], rational normal scrolls [MR13], Grassmannians [CMR15], some flag varieties [CMR15, CHW], generic K3 surfaces [AFO], abelian surfaces [Bea16], Enriques surfaces [BN] and ruled surfaces [ACMR].

The question of whether every subvariety of projective space admits an Ulrich sheaf was first posed in [ESW03] and remains open. It was shown in [KMS17] that an affirmative answer is equivalent to the simultaneous solution of a large number of higher-rank Brill-Noether problems on nongeneric curves. In light of the fact that the varieties known to admit Ulrich sheaves are mostly ACM, a natural first step is to restrict the question to ACM varieties.

It is straightforward to check that if $\mathcal{E}$ is an Ulrich sheaf on X, then the restriction of $\mathcal{E}$ to a general linear section is Ulrich. The converse holds for linear sections of dimension 2 or greater (Lemma

[*]Michigan State University, East Lansing, Michigan. kulkarni@math.msu.edu
[†]Tufts University, Medford, Massachusetts. Yusuf.Mustopa@tufts.edu
[‡]University of Utah, Salt Lake City, Utah. ian.shipman@gmail.com

3.1) but not linear sections of dimension 1 (e.g. Remark 3.6). In addition, Ulrich sheaves on 1-dimensional linear sections have very recently been used by Faenzi and Pons-Llopis to show that most ACM varieties are of wild representation type [FPL]. All this suggests a natural enlargement of the class of Ulrich sheaves whose existence problem may be more tractable.

Definition. Let $\mathcal{E}$ be a reflexive sheaf on a projective variety $X \subseteq \mathbb{P}^N$, and let $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. We say that $\mathcal{E}$ is $\delta-$*Ulrich* (with respect to $\mathcal{O}(1)$) if there exists a smooth 1-dimensional linear section Y of X such that the restriction $\mathcal{E}|_Y$ is Ulrich with respect to $\mathcal{O}_Y(1)$.

Perhaps the simplest examples of $\delta-$Ulrich sheaves that are not Ulrich are constructed in Theorem 2.2.5 of [OSS11]—they are nontrivial rank-2 bundles on $\mathbb{P}^2$ whose restriction to a general line is trivial. By definition they are $\delta-$Ulrich with respect to $\mathcal{O}_{\mathbb{P}^2}(1)$, but their nontriviality means they are not Ulrich. Though we are not aware of a complete characterization of sheaves on $\mathbb{P}^2$ that are $\delta-$Ulrich with respect to $\mathcal{O}_{\mathbb{P}^2}(1)$, Theorem 17 of [Jar06] implies they are all semistable instanton sheaves in the sense of [Jar06]. The connection with instanton sheaves is discussed in Section 4.1.

While $\delta-$Ulrich sheaves are a strict generalization of Ulrich sheaves, their existence is not immediate; it is worth mentioning that the examples we present in this paper (specifically, in Remarks 3.6 and 4.5) are on varieties already known to admit Ulrich sheaves. Our main result, which is implied by Theorem 2.7, is the following:

Theorem A. *Let $X \subseteq \mathbb{P}^N$ be a normal ACM variety. Then X admits a $\delta-$Ulrich sheaf.*

The $\delta-$Ulrich condition for a sheaf $\mathcal{F}$ on X can be rephrased as saying that if $\pi : X \to \mathbb{P}^n$ is a general finite linear projection, the direct image $\pi_*\mathcal{F}$ restricts to a trivial vector bundle on a general line $\ell \subseteq \mathbb{P}^n$, so to construct a $\delta-$Ulrich sheaf on X amounts to finding a reflexive sheaf $\mathcal{E}$ on $\mathbb{P}^n$ and a line $\ell \subseteq \mathbb{P}^n$ such that $\mathcal{E}$ is a $\pi_*\mathcal{O}_X-$module and $\mathcal{E}|_\ell$ is a trivial vector bundle on ℓ. It suffices to carry this out this construction on an open subset of $\mathbb{P}^n$ whose complement is of codimension at least 2.

Lemma 2.4 implies that if $\pi : X \to \mathbb{P}^n$ is a finite linear projection, then there are open affine subsets $V_1, V_2 \subseteq \mathbb{P}^n$ and polynomials $p_i(z_i) \in \mathcal{O}_{V_i}[z_i]$ such that the complement of $V_1 \cup V_2$ is of codimension 2 and $\pi_*\mathcal{O}_X|_{V_i} \cong \mathcal{O}_{V_i}[z_i]/(p_i(z_i))$. Our strategy for proving Theorem A begins with constructing for $i = 1, 2$ a locally Cohen-Macaulay sheaf $\mathcal{E}_i$ on V_i which admits the structure of a $\pi_*\mathcal{O}_X|_{V_i}$-module. What allows us to do this is the notion of a *characteristic morphism* of (sheaves of) algebras. Such morphisms generalize algebra homomorphisms in the sense that they respect the Cayley-Hamilton theorem; see Section 1.1 for details, as well as [KMS]. Although we are not aware of any earlier work on characteristic morphisms as such, we were inspired by the use of characteristic polynomials in [Pap00]. For similar ideas in the context of invariant theory, see [Pro87].

It is not obvious that $\mathcal{E}_1$ and $\mathcal{E}_2$ glue together to form a $\pi_*\mathcal{O}_X|_{V_1 \cup V_2}-$module. However, the special characteristic morphism we construct in Proposition 2.5 ensures that the restrictions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to a general line $\ell \subseteq V_1 \cup V_2$ glue together to form an Ulrich sheaf for the restriction $\pi^{-1}(\ell) \to \ell$ of π. The $\delta-$Ulrich sheaf we produce is an algebraization of a sheaf on the formal neighborhood of ℓ which comes from gluing completions of $\mathcal{E}_1$ and $\mathcal{E}_2$ along this neighborhood (Lemma 2.6 and Theorem 2.7).

Even though it is not used explicitly, the central concept underlying the proof of Proposition 2.5 is that of the Roby-Clifford algebra R_F of a degree-d homogeneous form F over a field $\mathbf{k}$. This was introduced by Roby in [Rob69], and it directly generalizes the classical Clifford algebra of a quadratic form, as R_F satisfies a similar, higher-degree universal property (see Remark 1.2). It is shown in [VdB87] that Ulrich sheaves on the cyclic covering hypersurface $\{w^d = F\}$ correspond to finite-dimensional R_F-modules, and a more refined correspondence involving the natural $\mathbb{Z}/d\mathbb{Z}$-grading on R_F is used in [BHS88] to construct Ulrich sheaves on hypersurfaces. The latter construction uses the $\mathbb{Z}/d\mathbb{Z}$*-graded tensor product* of modules over Roby-Clifford algebras (see Section 1.2) to construct an Ulrich sheaf over the zero locus of the "generic homogeneous form of degree d which is a sum of s monomials." Our proof of Proposition 2.5 uses $\mathbb{Z}/d\mathbb{Z}$-graded tensor products to extend an algebraic object (the characteristic morphism) from the line $\ell \subset \mathbb{P}^n$ to all of $\mathbb{P}^n$.

We can say more about $\delta-$Ulrich sheaves when X is a normal ACM surface. It is immediate from the definition that $\delta-$Ulrich sheaves on normal ACM surfaces are locally Cohen-Macaulay, a necessary condition for being Ulrich. As mentioned earlier, the sheaves on $\mathbb{P}^2$ which are $\delta-$Ulrich with respect to $\mathcal{O}_{\mathbb{P}^2}(1)$ are semistable instanton sheaves, so in general, $\delta-$Ulrich sheaves on a surface have the property that their direct image under a finite linear projection is a semistable instanton sheaf (Proposition 4.1). We show that the intermediate cohomology module $H^1_*(\mathcal{E})$ satisfies the Weak Lefschetz property (Proposition 4.12); moreover, the maximum value of the Hilbert function of $H^1_*(\mathcal{E})$ is $h^1(\mathcal{E}(-1))$.

A substantial difference between Ulrich and $\delta-$Ulrich sheaves is that the former are globally generated, while the latter need not have any global sections at all (compare Remark 3.6). However, a $\delta-$Ulrich sheaf $\mathcal{E}$ on X is Ulrich if and only if it has $\deg(X)\cdot \mathrm{rk}(\mathcal{E})$ global sections (see Proposition 3.2). If we replace $\mathcal{O}_X(1)$ by a potentially high twist, we have enough control on the cohomology to obtain the following result.

Theorem B. *If $X \subseteq \mathbb{P}^N$ is a smooth ACM surface, there exists $k > 0$ such X admits a $\delta-$Ulrich sheaf with respect to $\mathcal{O}_X(k)$ possessing a global section.*

This theorem follows from a more precise statement. If $\mathcal{E}$ is a $\delta-$Ulrich sheaf on X, consider the quantity

$$\alpha(\mathcal{E}) = h^0(\mathcal{E})/\deg(X)\mathrm{rk}(\mathcal{E})$$

Our earlier observation can be rephrased as saying that $\mathcal{E}$ is Ulrich if and only if $\alpha(\mathcal{E}) = 1$. Theorem B is proved by exhibiting a sequence of sheaves $\{\mathcal{E}_m\}_m$ where $\mathcal{E}_m$ is δ-Ulrich with respect to $\mathcal{O}_X(2^m)$ and such that $\lim_{m\to\infty} \alpha(\mathcal{E}_m) = 1$.

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Notation and Conventions

Our base field $\mathbf{k}$ is algebraically closed of characteristic zero. All open subsets are Zariski-open. If R is a ring we use the notation $R\{t_1, \ldots, t_n\}$ for the free R-module with basis $t_1, \ldots, t_n$.

1 Preliminaries

In this section, we collect the algebraic prerequisites for the proof of Theorem A. Throughout, R denotes a commutative $\mathbf{k}-$algebra and A denotes a commutative $R-$algebra which is free of rank $d \geq 2$ as an $R-$module.

1.1 Roby Modules and Characteristic Morphisms

Definition 1.1. Let M, W be free $R-$modules and let $F \in \mathrm{Sym}^\bullet_R(M^\vee)$ be a homogeneous form of degree $e \geq 2$. An $R-$module morphism $\phi : M \to \mathrm{End}_R(W)$ is an $F-$Roby module if for all $m \in M$ we have

$$\phi(m)^e = F(m) \cdot \mathrm{id}_W$$

where $F(m)$ is the image of $m^{\otimes e}$ under the symmetric map $M^{\otimes e} \to R$ associated to F. If, in addition, W is a $\mathbb{Z}/e\mathbb{Z}$-graded $R-$module and $\phi(m)$ is a degree-1 endomorphism for $0 \neq m \in M$, we say that ϕ is a *graded* $F-$Roby module.

Remark 1.2. The terminology can be explained as follows. If ϕ is an $F-$Roby module, the induced $R-$algebra morphism $T^\bullet_R(M) \to \mathrm{End}_R(W)$ annihilates $\{\phi(m)^e - F(m) : m \in M\}$, and therefore descends to a morphism $R_F \to \mathrm{End}_R(W)$, where

$$R_F := T^\bullet_R(M)/\langle \phi(m)^e - F(m) : m \in M\rangle$$

is the Roby-Clifford algebra of F (see [Rob69]). Conversely, given an $R-$algebra morphism $R_F \to \mathrm{End}_R(W)$, we recover an $F-$Roby module by composing with the natural injection $M \hookrightarrow R_F$.

Example 1.3. We recall a construction from [Chi78] which will be used in the proof of Proposition 2.5. Let $M = R\{x_1, \ldots, x_n\}$ and suppose that $y_1, \ldots, y_n$ is the dual basis of $M^\vee$. Consider a monomial $F = y_{i_1} y_{i_2} \ldots y_{i_e} \in \mathrm{Sym}^e_R(M^\vee)$ and put $W = R\{w_1, \ldots, w_e\}$. Then there is a natural, $\mathbb{Z}/e\mathbb{Z}-$graded $F-$Roby module $\phi : M \to \mathrm{End}_R(W)$ given by

$$\phi(x_i)(w_j) = \begin{cases} w_{j+1} & i = i_j, \\ 0 & \text{otherwise,} \end{cases}$$

where the indices on the elements $w_1, \ldots, w_e$ are taken modulo e, and $\deg(w_i) = i$ for all i.

Definition 1.4. If A is an associative $R-$algebra whose underlying $R-$module is of finite rank d, the *characteristic polynomial* of A is

$$\chi_A(t, a) := \det(tI - \rho_A(a)) = \sum_{j=0}^{d} (-1)^j \mathrm{tr}(\wedge^j \rho_A(a)) \cdot t^{d-j}$$

where $\rho_A : A \to \mathrm{End}_R(A)$ is the regular representation of A.

Observe that $\chi_A(t, a)$ is a degree-d element of $\mathrm{Sym}^\bullet_R(A^\vee) \otimes_R R[t] \cong \mathrm{Sym}^\bullet_R(A^\vee \oplus R\{t\})$. Also, if B is an $R-$algebra, then for any $a \in A$ and $b \in B$ we have that $\chi_A(b, a)$ is a well-defined element of B.

Example 1.5. Consider the $R-$algebra $A = R^{\times d}$. We identify $R^{\times d} = R\{e_1, \ldots, e_d\}$ where $\{e_i\}$ is the standard basis of idempotents. Under the regular representation we have $\rho_A(a_1, \ldots, a_d) = \mathrm{diag}(a_1, \ldots, a_d)$ and therefore $\chi_A(t, a_1, \ldots, a_d) = (t - a_1) \cdots (t - a_d)$. It folows that

$$\chi_A(t) = (t - x_1) \cdots (t - x_d)$$

where $x_1, \ldots, x_d$ is the dual basis to $e_1, \ldots, e_d$.

We record the following elementary properties, which will be used in the sequel.

Lemma 1.6. *Let B be an $R-$algebra.*

(i) If B is commutative and free of finite rank as an $R-$module, then χ_A is taken to $\chi_{A\otimes_R B}$ under the natural map

$$\mathrm{Sym}_R^\bullet(A^\vee)[t] \to \mathrm{Sym}_B^\bullet((A\otimes_R B)^\vee)[t]$$

induced by the base-change map $A^\vee \to (A\otimes_R B)^\vee = \mathrm{Hom}_B(A\otimes_R B, B)$.

(ii) If $B \to C$ is an embedding of $R-$algebras which are both free of the same finite rank, then χ_B is the image of χ_C under the natural morphism

$$\mathrm{Sym}_R^\bullet(C^\vee)[t] \to \mathrm{Sym}_R^\bullet(B^\vee)[t].$$

□

If $\phi : A \to B$ is a morphism of $R-$algebras, the Cayley-Hamilton theorem implies that

$$\chi_A(\phi(a), a) = \phi(\chi_A(a, a)) = 0$$

for all $a \in A$. The more general notion that follows is a key ingredient in our construction of $\delta-$Ulrich sheaves.

Definition 1.7. If B is an $R-$algebra, an $R-$module morphism $\phi : A \to B$ is a *characteristic morphism* if $\chi_A(\phi(a), a) = 0$ for all $a \in A$.

Remark 1.8. The notion of a characteristic morphism is strictly more general than that of an $R-$algebra morphism. If $A = R\{e_1, e_2\}$ is the $R-$algebra generated by the orthogonal idempotents e_1 and e_2, then for any $a, b \in R$ satisfying $a + b \neq 0$, the map $\phi : A \to \mathrm{Mat}_2(R)$ defined by

$$\phi(e_1) = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \ \ \phi(e_2) = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$$

is a characteristic morphism, but not an $R-$algebra morphism.

We now turn to the sheaf-theoretic formulations of these concepts. For the remainder of this subsection, Y denotes a smooth irreducible quasi-projective variety, $\mathcal{A}$ denotes a sheaf of $\mathcal{O}_Y-$algebras which is locally free of rank $d \geq 2$, and W denotes a finite-dimensional $\mathbf{k}$-vector space. For a sheaf $\mathcal{F}$ on Y, we denote the stalk of $\mathcal{F}$ at a point $y \in Y$ by $\mathcal{F}_y$ and the $\mathbf{k}(Y)-$vector space of rational sections of $\mathcal{F}$ by $\mathcal{F}(Y)$.

Definition 1.9. If $\mathcal{B}$ is a coherent sheaf of $\mathcal{O}_Y-$algebras, a $\mathcal{O}_Y-$linear morphism $\phi : \mathcal{A} \to \mathcal{B}$ is a *characteristic morphism* if for each $y \in Y$, the $\mathcal{O}_{Y,y}-$module morphism $\phi_y : \mathcal{A}_y \to \mathcal{B}_y$ is a characteristic morphism.

The following observation will be used later.

Lemma 1.10. *$\phi : \mathcal{A} \to \mathcal{B}$ is a characteristic morphism if and only if the induced $\mathbf{k}(Y)-$linear map $\phi_{\mathbf{k}(Y)} : \mathcal{A}(Y) \to \mathcal{B}(Y)$ is a characteristic morphism.* □

If $\rho_A : \mathcal{A} \to \mathcal{E}nd(\mathcal{A})$ is the regular representation of $\mathcal{A}$, then since $\mathrm{tr}(\wedge^j \rho_{\mathcal{A}})$ is a global section of $\mathrm{Sym}^j(\mathcal{A}^\vee)$ for each j, there exists a global characteristic polynomial $\chi_{\mathcal{A}} \in H^0(\mathrm{Sym}^\bullet(\mathcal{A}^\vee \oplus \mathcal{O}_Y\{t\}))$.

Definition 1.11. An $\mathcal{O}_Y-$linear morphism $\psi : \mathcal{A} \oplus \mathcal{O}_Y\{T\} \to \mathrm{End}(W \otimes \mathcal{O}_Y)$ is a $\chi_{\mathcal{A}}-$*Roby module* if for each $y \in Y$, the $\mathcal{O}_{Y,y}-$module morphism $\psi_y : \mathcal{A}_y \oplus \mathcal{O}_{Y,y}\{T\} \to \mathrm{End}(W \otimes \mathcal{O}_{Y,y})$ is a $\chi_{\mathcal{A}}-$Roby module in the sense that for all $a \in \mathcal{A}_y$ and all $r \in \mathcal{O}_{Y,y}$ we have

$$\psi(a, rT)^d = \chi_{\mathcal{A}}(a, r) \cdot \mathrm{Id}.$$

If W is $\mathbb{Z}/d\mathbb{Z}-$graded and $\psi(a, rT)$ is a degree-1 endomorphism for all local sections a, r then ψ is a *graded* $\chi_{\mathcal{A}}-$*Roby module.*

If ψ is a graded $\chi_{\mathcal{A}}$-Roby module as above, then $\psi(T)$ is globally defined, and $\psi(T)^d = \mathrm{Id}$ since $\chi_{\mathcal{A}}$ is monic in t. In particular, $\psi(T)$ is invertible in $\mathrm{End}(W \otimes \mathcal{O}_Y)$.

Lemma 1.12. *Let $\psi : \mathcal{A} \oplus \mathcal{O}_Y\{T\} \to \mathrm{End}(W \otimes \mathcal{O}_Y)$ be a graded Roby $\chi_{\mathcal{A}}-$module. Then the morphism $C_\psi : \mathcal{A} \to \mathrm{End}(W \otimes \mathcal{O}_Y)$ defined by the composition*

$$\mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathcal{O}_Y\{T\} \xrightarrow{-\psi} \mathrm{End}(W \otimes \mathcal{O}_Y) \xrightarrow{\cdot \psi(T)^{-1}} \mathrm{End}(W \otimes \mathcal{O}_Y)$$

is a characteristic morphism.

Proof. By Lemma 1.10 it suffices to consider a field extension $K/\mathbf{k}$ and a $d-$dimensional commutative $K-$algebra A in place of $\mathcal{O}_Y$ and $\mathcal{A}$, respectively. Put $W_K = W \otimes_{\mathbf{k}} K$. Let $\chi_A = \chi_A(t)$ be the characteristic polynomial of A, and let $\psi : A \oplus K\{T\} \to \mathrm{End}_K(W_K)$ be a graded χ_A-Roby module. Then ψ corresponds to an element $\psi^\vee$ of $\mathrm{End}_K(W_K) \otimes (A^\vee \oplus K\{t\})$ whose $d-$th power

$$(\psi^\vee)^d \in \mathrm{End}_K(W_K) \otimes \mathrm{Sym}^d(A^\vee \oplus K\{t\}) \cong \mathrm{Hom}(W_K, W_K \otimes \mathrm{Sym}^d(A^\vee \oplus K\{t\}))$$

is equal to $1_W \otimes \chi_A$.

Consider the graded $S = \mathrm{Sym}^\bullet(A^\vee)[t, w]/(w^d - \chi_A)$-module $M = W_K \otimes_K \mathrm{Sym}^\bullet(A^\vee)[t]$ on which w acts by $\psi^\vee$ (and $A^\vee, t$ act in the obvious way). Now, M is a graded maximal Cohen-Macaulay S-module, generated in degree zero–a graded Ulrich module, in fact. Since $\mathrm{Sym}^\bullet(A^\vee)[w]$ is a standard-graded polynomial subring of S over which S is finite and flat, it follows that M is graded-free over $\mathrm{Sym}^\bullet(A^\vee)[w]$ and generated in degree zero. Consequently, the map

$$W_K \otimes_K \mathrm{Sym}^\bullet(A^\vee)[w] \to M$$

is an isomorphism. We aim to compute the action of t in terms of the action of w and $A^\vee$. We can write

$$\psi^\vee = \psi_0^\vee + \psi(T) \otimes t, \quad \psi_0^\vee \in \mathrm{End}_K(W_K) \otimes A^\vee.$$

So if $m \in M$ we have

$$wm = \psi_0^\vee m + t\psi(T)m.$$

As we observed earlier, $\psi(T)$ is invertible. Replacing m by $\psi(T)^{-1}m$ in the previous equation, we have that

$$tm = w\psi(T)^{-1}m - \psi_0^\vee \psi(T)^{-1}m.$$

Let $\overline{M}$ be the graded $\mathrm{Sym}^\bullet(A^\vee)[t]/(\chi_A(t))$-module obtained from reducing the $S-$module structure of M modulo w. Then $\overline{M}$ is graded-free over $\mathrm{Sym}^\bullet(A^\vee)$, and it is generated in degree zero. Now, t acts on $\overline{M}$ by $-\psi_0^\vee \psi(T)^{-1}$. Since $\chi_A(t)$ acts by zero on $\overline{M}$, the map $A \to \mathrm{End}_K(W_K)$ corresponding to $-\psi_0^\vee \psi(T)^{-1}$ is a characteristic morphism. This map is C_ψ, so we see that C_ψ is a characteristic morphism. □

Example 1.13. The following construction will be used in the proof of Lemma 2.1. Once again, we consider $A = R^{\times d} = R\{e_1, \dots, e_d\}$. From Example 1.5 we see that $\chi_A(t) = \prod_{i=1}^d (t - x_i)$ where $\{x_i\}$ is the dual basis to $\{e_i\}$. There is a natural graded χ_A-Roby module $\phi : A \oplus R\{T\} \to \mathrm{End}_R(R\{w_1, \dots, w_d\})$ defined by

$$\phi(T)(w_i) = w_{i+1}, \quad \phi(e_i)(w_j) = \begin{cases} -w_{j+1} & i = j+1, \\ 0 & i \neq j+1, \end{cases}$$

where the indices on the w_i are taken modulo d and $\deg(w_i) = i$. Since

$$\phi(rT + \sum_{i=1}^r a_i e_i)(w_j) = (r - a_{j+1})w_{j+1}$$

we see by iteration that ϕ is indeed a χ_A-Roby module. Let us compute C_ϕ. We have

$$C_\phi(e_i)(w_j) = -\phi(e_i)(\phi(T)^{-1}(w_j)) = -\phi(e_i)(w_{j-1}) = \delta^i_j w_j.$$

So, better than simply being a characteristic morphism we see that C_ϕ is an algebra morphism, equipping W with the structure of a free $R^{\times d}$-module. However, note that the construction of this module implicitly relied on a cyclic ordering on the idempotents $e_\bullet \in A$.

Remark 1.14. We note that the formation of C_ϕ is functorial. More precisely, suppose that $\phi : \mathcal{A} \oplus \mathcal{O}\{T\} \to \mathrm{End}(W) \otimes \mathcal{O}$ is a χ-Roby morphism. If $W' \subset W$ is an invariant subspace in the sense that for any local section $a + rT$ of $\mathcal{A} \oplus \mathcal{O}\{T\}$, the action of $\phi(a, rT)$ on $W \otimes \mathcal{O}$ sends $W' \otimes \mathcal{O}$ into itself, then the action of $\mathcal{A}$ via C_ϕ will also send $W' \otimes \mathcal{O}$ into itself. This equips $W' \otimes \mathcal{O}$ and $W/W' \otimes \mathcal{O}$ with the structures of χ-Roby modules and characteristic modules, respectively.

1.2 $\mathbb{Z}/d\mathbb{Z}-$Graded Tensor Products

The following notion, which was first applied to the study of Roby modules in [Chi78], is required for the proof of Proposition 2.5. The proof is essentially that of Theorem 3.1 in [BHS88].

Proposition-Definition 1.15. *Let M be a free $R-$module and $F_1, F_2 \in \mathrm{Sym}^d_R(M^\vee)$ homogeneous forms. Suppose that for each $i = 1, 2$ we have a graded F_i-Roby module $\phi_i : M \to \mathrm{End}_R(W_i)$, where W_1, W_2 are $\mathbb{Z}/d\mathbb{Z}-$graded $R-$modules. Then the morphism $\phi : M \to \mathrm{End}_R(W_1 \otimes W_2)$ defined by*

$$\phi(m)(w_1 \otimes w_2) = \phi_1(m)(w_1) \otimes w_2 + \xi^{\deg(w_1)} w_1 \otimes \phi_2(m)(w_2)$$

is a graded $F_1 + F_2-$Roby module, where $W_1 \otimes W_2$ is graded by $\deg(w_1 \otimes w_2) = \deg(w_1) + \deg(w_2)$ for homogeneous elements $w_i \in W_i$. We denote this morphism by $\phi = \phi_1 \widehat{\otimes}_\xi \phi_2$.

2 Construction of δ-Ulrich Sheaves

We now take up the proof of Theorem A in earnest. As our first step, we use Lemma 1.12 to produce a type of enhanced Ulrich sheaf for any finite covering of $\mathbb{P}^1$.

Lemma 2.1. *Let C be a smooth curve and let $f : C \to \mathbb{P}^1$ be a morphism of degree $d \geq 2$ Then there exists a graded $\chi_{f_*\mathcal{O}_C}$-Roby module ψ whose associated characteristic morphism C_ψ is an $f_*\mathcal{O}_C-$module morphism.*

Proof. Let $K(C)/K(\mathbb{P}^1)$ be the field extension corresponding to f. Since the extension is separated, it has the form $K(C) \cong K(\mathbb{P}^1)[z]/(p(z))$ for some polynomial $p(z)$. Let L be the splitting field of $p(z)$ and $g : D \to \mathbb{P}^1$ the map of curves corresponding to $L/K(\mathbb{P}^1)$. Then $C \times_{\mathbb{P}^1} D$ has d components, each of which is isomorphic to D. So we have a diagram

$$\begin{array}{ccccc} \sqcup_{i=1}^d D & & & & \\ & \searrow^{\eta} & & & \\ & & C \times_{\mathbb{P}^1} D & \to & D \\ & & \downarrow & & \downarrow g \\ & & C & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

where η is the normalization of $C \times_{\mathbb{P}^1} D$. Let $\mathcal{A} = f_*\mathcal{O}_C$, $\mathcal{B} = g_*\mathcal{O}_D$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$. Since C and D are reduced curves, they are locally CM, so $\mathcal{A}$ and $\mathcal{B}$ are locally free as $\mathcal{O}-$modules; in particular, $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}$ is locally free as a $\mathcal{B}-$module. Also, η is induced by a $\mathcal{B}-$module morphism $\widetilde{\eta} : \mathcal{A} \otimes_{\mathcal{O}} \mathcal{B} \to \mathbf{k}^{\times d} \otimes_{\mathbf{k}} \mathcal{B}$.

Let $\chi(t) \in \mathrm{Sym}^\bullet_{\mathcal{O}}(\mathcal{A}^\vee)[t]$, $\widetilde{\chi}(t) \in \mathrm{Sym}^\bullet_{\mathcal{B}}((\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B})^\vee)[t]$, and $\widetilde{\chi}_s(t) \in \mathrm{Sym}^\bullet_{\mathcal{B}}((\mathcal{B}^\vee \otimes_{\mathbf{k}} \mathbf{k}^{\times d})^\vee)[t]$ be the characteristic polynomials of $\mathcal{A}$ over $\mathcal{O}$, $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}$ over $\mathcal{B}$, and $\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}$ over $\mathcal{B}$ respectively. Then (according to Lemma 1.6) under the natural maps

$$\mathrm{Sym}^\bullet_{\mathcal{B}}((\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d})^\vee)[t] \to \mathrm{Sym}^\bullet_{\mathcal{B}}((\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B})^\vee)[t]$$
$$\mathrm{Sym}^\bullet_{\mathcal{O}}(\mathcal{A}^\vee)[t] \to \mathrm{Sym}^\bullet_{\mathcal{B}}((\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B})^\vee)[t]$$

we see that $\widetilde{\chi}_s(t)$ maps to $\widetilde{\chi}(t)$ and $\chi(t)$ maps to $\widetilde{\chi}(t)$. This means that if a and r are local sections of $\mathcal{A}$ and $\mathcal{O}$, respectively, then $\widetilde{\chi}_s(a, rt) = \chi(a, rt)$.

As in Example 1.13, there is a natural graded $\widetilde{\chi}_s(t)$-Roby module $\phi : (\mathbf{k}^{\times d} \otimes_{\mathbf{k}} \mathcal{B}) \oplus \mathcal{B}\{T\} \to \mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d})$ defined by

$$\phi(T)(e_j) = e_{j+1}, \quad \psi(e_i)(e_j) = \begin{cases} -e_{j+1} & i = j+1 \\ 0 & i \neq j+1 \end{cases}$$

where the indices of the standard idempotents e_i are taken modulo d. Clearly, ϕ is a Roby module for the characteristic polynomial $\widetilde{\chi}$ for $\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}$ over $\mathcal{B}$. Moreover, one can verify that C_ϕ is an algebra morphism.

By [ESW03], there is an Ulrich sheaf $\mathcal{E}$ for D over $\mathbb{P}^1$, which we view as a $\mathcal{B}$-module on $\mathbb{P}^1$. Note that there is an algebra morphism

$$\mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d}) \to \mathrm{End}_{\mathcal{O}}(\mathcal{E} \otimes \mathbf{k}^{\times d})$$

defined by tensoring a map with $\mathcal{E}$ over $\mathcal{B}$. We can then obtain a map

$$\widetilde{\phi} : \mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d} \oplus \mathcal{B} \cdot T \to \mathrm{End}(\mathcal{B} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}) \to \mathrm{End}_{\mathcal{O}}(\mathcal{E} \otimes_{\mathbf{k}} \mathbf{k}^{\times d}) = \mathrm{End}(W \otimes \mathbf{k}^{\times d} \otimes \mathcal{O})$$

where we choose a trivialization $\mathcal{E} \cong W \otimes \mathcal{O}$ as $\mathcal{O}$-modules. Now this induces a χ-Roby module structure on $W \otimes \mathbf{k}^{\times d} \otimes \mathcal{O}$. Let $\psi : \mathcal{A} \oplus \mathcal{O}\{T\} \to \mathrm{End}(W \otimes \mathbf{k}^{\times d}) \otimes \mathcal{O}$ be the restriction of $\widetilde{\phi}$ to $\mathcal{A} \oplus \mathcal{O}\{T\} \subset \mathcal{B} \otimes \mathbf{k}^{\times d} \oplus \mathcal{B}\{T\}$, where $\mathcal{A} \to \mathcal{B} \otimes \mathbf{k}^{\times d}$ is the map $\mathcal{A} \otimes 1 \to \mathcal{A} \otimes \mathcal{B} \xrightarrow{\widetilde{\eta}} \mathcal{B} \otimes \mathbf{k}^{\times d}$.

Suppose that a, r are local sections of $\mathcal{A}$ and $\mathcal{O}$ respectively. Then we know that

$$\phi(a, r)^d = \widetilde{\chi}_s(a, r) \cdot \mathrm{id} \in \mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d})$$

Now $\widetilde{\chi}_s(a, r) = \chi(a, r) \in \mathcal{O}$. Since the morphism $\mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d}) \to \mathrm{End}(W \otimes \mathbf{k}^{\times d}) \otimes \mathcal{O}$ is $\mathcal{O}$-linear, ψ is a χ-Roby module. Finally, we must check that C_ψ is a morphism. Now, $C_\phi = R_{\phi(T)^{-1}}\phi$ is a morphism, where $R_{\phi(T)^{-1}}$ is right multiplication by $\phi(T)^{-1}$. Composing C_ϕ with the map $\mathrm{End}_{\mathcal{B}}(\mathcal{B} \otimes \mathbf{k}^{\times d}) \to \mathrm{End}(W \otimes \mathbf{k}^{\times d}) \otimes \mathcal{O}$, we obtain $R_{\widetilde{\phi}(T)^{-1}}\widetilde{\phi}$ and this is still a morphism. Now, if we restrict this morphism to $\mathcal{A}$ we obtain $C_\psi = R_{\psi(T)^{-1}}\psi$ since $\psi(T) = \widetilde{\phi}(T)$ and ψ is the restriction of $\widetilde{\phi}$ to $\mathcal{A}$. □

Definition 2.2. Let W be a vector space, and let $F^\bullet$ be an increasing filtration on W. A *filtered pseudomorphism* $\phi : \mathcal{A} \to \mathrm{End}(W \otimes \mathcal{O}_{\mathbb{P}^n})$ is a characteristic morphism satisfying the following properties:

(i) The image of ϕ is contained in the algebra $\mathrm{End}_{F^\bullet}(W \otimes \mathcal{O}_{\mathbb{P}^n})$ of endomorphisms preserving $F^\bullet$.

(ii) The induced map $\phi_{F^\bullet} : \mathcal{A} \to \Pi_i \mathrm{End}(F^{i+1}W/F^iW \otimes \mathcal{O}_{\mathbb{P}^n})$ is an $\mathcal{O}_{\mathbb{P}^n}$-algebra morphism.

Our $\delta-$Ulrich sheaf will come from a characteristic morphism that restricts to a filtered pseudomorphism on a 1-dimensional linear section.

Definition 2.3. If R is a commutative ring, a finitely generated $R-$algebra A is said to be *monogenic* if there exists a monic polynomial $p[z] \in R[z]$ such that $A \cong R[z]/\langle p(z)\rangle$. If Y is a quasi-projective variety and $U \subseteq Y$ is open, a coherent sheaf $\mathcal{A}$ of $\mathcal{O}_Y$-algebras is said to be *monogenic on* U if $\mathcal{A}|_U$ is a monogenic $\mathcal{O}_U-$algebra.

Lemma 2.4. *Let $X \subseteq \mathbb{P}^N$ be a subvariety of dimension n which is regular in codimension 1, where $2 \leq n \leq N-2$. Then for a general finite linear projection $\pi : X \to \mathbb{P}^n$, there are affine open sets $V_1, V_2 \subseteq \mathbb{P}^n$ satisfying the following conditions:*

(i) $\pi_*\mathcal{O}_X$ *is monogenic on* V_1 *and* V_2.

(ii) $V_1 \cup V_2$ *contains a line* ℓ *such that* $X_\ell := \pi^{-1}(\ell)$ *is smooth and contained in the regular locus* X^{reg}.

Proof. Consider the space of triples $P \subset X \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1) \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n)$ defined as the closure of

$$P^o = \{(x, \Lambda', \Lambda) : x \in X^{\text{reg}},\ x \in \Lambda,\ \Lambda' \subset \Lambda,\ \dim(T_xX \cap T_x\Lambda) > 1\}.$$

Let $P' \subset X \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ be the image of P under projection. For a general (x, Λ') in P', we have $x \notin \Lambda'$ and therefore Λ is the projective span of Λ' and x. So $\dim(P) = \dim(P')$. Let Q be the image of P in $X \times \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n)$. Then $P \to Q$ is generically a projective space bundle whose fibers have dimension $N-n$. So $\dim(P) = \dim(Q) + N - n$. We will compute the dimension of Q, using the projection to X. Let $x \in X^{\text{reg}}$. Then the fiber of Q over x is birationally isomorphic to the set of pairs

$$\{(\alpha, \Lambda) : \alpha \in \mathrm{Gr}(T_xX, 2), x \in \Lambda \in \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n), \alpha \subset T_x\Lambda\}.$$

This set of pairs is a $\mathrm{Gr}(N-2, N-n-2)-$bundle over $\mathrm{Gr}(T_xX, 2) \cong \mathrm{Gr}(n, 2)$. So it has dimension $2(n-2) + n(N-n-2)$. Hence we see that $\dim(Q) = n + 2(n-2) + n(N-n-2)$. Finally we deduce that

$$\dim(P') = N + 2(n-2) + n(N-n-2).$$

In what follows, we denote the linear span of $\Lambda' \in \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ and $x \in X$ by Λ'_x. If $P' \to \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ is dominant, then a general fiber has dimension

$$N + 2(n-2) + n(N-n-2) - (N-n)(n+1) = n-4.$$

This means that for a general $(N-n-1)-$plane Λ', we have $\dim(T_x\Lambda'_x \cap T_xX) \leq 1$ for all x away from a subset of X^{reg} having codimension at least 4. If $P' \to \mathbb{G}\mathrm{r}(\mathbb{P}^N, N-n-1)$ is not dominant, then a general $(N-n-1)-$plane Λ' would have the property that for any $x \in X^{\text{reg}}$, $\dim(T_x\Lambda'_x \cap T_xX) \leq 1$. In either case, for a general $(N-n-1)-$plane Λ', we have that $\Lambda' \cap X = \varnothing$, Λ'_x is transverse to X at a general point, and away from a locus of codimension at least two, $T_x\Lambda'_x \cap T_xX$ is at most one dimensional. Fixing such a Λ' for the rest of the proof, we define Z to be the union of X^{sing} and the set of all $x \in X$ for which at least one of these properties fails. Our discussion thus far implies that Z is of codimension at least 2 in X.

Let $\pi : X \to \mathbb{P}^n$ be the finite projection associated to Λ', and let $\mathcal{A} := \pi_*\mathcal{O}_X$ be the associated locally free sheaf of $\mathcal{O}_{\mathbb{P}^n}$-algebras. Given $p \in \mathbb{P}^n \setminus \pi(Z)$ and $x \in \pi^{-1}(p)$, we see that since $T_x\Lambda'_x \cap T_xX$ is at most one-dimensional, the cotangent space to $\pi^{-1}(p)$ at x is at most one-dimensional. Hence $\mathcal{A}|_p$ is a monogenic $\mathcal{O}_{\mathbb{P}^n,p}-$algebra for all $p \in \mathbb{P}^n \setminus \pi(Z)$ (this will be used shortly).

Consider an affine open set $V \subset \mathbb{P}^n \setminus \pi(Z)$. Let $y \in V$ be some point and let $z \in \mathcal{A}(V)$ be an element such that $z|_y$ is a generator for $\mathcal{A}|_y$. Then there is a polynomial $p(z)$ (the characteristic polynomial of z) such that the map $\mathcal{O}_V[z]/\langle p(z)\rangle \to \mathcal{A}|_V$ is an isomorphism away from a divisor $D \subset V$. Put $V_1 = V \setminus D$. Note that V_1 is affine and $\mathcal{A}$ is monogenic on V_1.

Let $\ell \subset \mathbb{P}^n$ be a line which avoids $\pi(Z)$, has nonempty intersection with V_1, and is such that $\pi^{-1}(\ell)$ is smooth. Let $y_1, \ldots, y_r$ be the members of $\ell \cap (\mathbb{P}^n \setminus V_1)$, and let $V' \subset \mathbb{P}^n \setminus \pi(Z)$ be an affine open set that contains all of the y_i. For each i, the algebra $\mathcal{A}|_{y_i}$ is monogenic, so we can fix a generator z_i of $\mathcal{A}|_{y_i}$. Since $\mathcal{A}(V') \to \prod_{i=1}^r \mathcal{A}|_{y_i}$ is surjective, there is an element $z \in \mathcal{A}(V')$ whose restriction to y_i is z_i. Now as before there is a polynomial $q(z)$ such that the map $\mathcal{O}_{V'}[z]/\langle q(z)\rangle \to \mathcal{A}|_V$ is an isomorphism away from a divisor $D' \subset V'$. By construction $y_1, \ldots, y_r \notin D'$. Put $V_2 = V' \setminus D'$. Then $\mathcal{A}$ is monogenic on V_2, and moreover $\ell \subset V_1 \cup V_2$. □

Proposition 2.5. *Let $X \subseteq \mathbb{P}^N$ be a normal ACM variety of dimension $n \geq 2$, and let $\pi : X \to \mathbb{P}^n$ be a finite linear projection. If $\ell \subseteq \mathbb{P}^n$ is a line, there exists a filtered vector space W and a characteristic morphism $\phi : \mathcal{A} \to \operatorname{End}(W \otimes \mathcal{O}_{\mathbb{P}^n})$ such that $\phi|_\ell$ is a filtered pseudomorphism.*

Proof. Let $x, y, z_2, \ldots, z_n$ be a coordinate system on $\mathbb{P}^n$ such that $\ell = V(z_2, \ldots, z_n)$. It is convenient to work with graded rings instead of schemes. So let us view $\mathbb{P}^n = \operatorname{Proj}(R)$ where $R = \mathbf{k}[x, y, z_2, \ldots, z_n]$ and $X = \operatorname{Proj}(S)$ where S is a graded Cohen-Macaulay R-algebra. Given that S is a free $R-$module, we fix a homogeneous basis $1 = \gamma_1, \ldots, \gamma_d$ for S as an R-module. Note that $\deg(\gamma_i) > 0$ for $i > 1$. Moreover $\ell = \operatorname{Proj}(\mathbf{k}[x, y])$. Let $\overline{S} = S/(z_2, \ldots, z_n)S$ and write $\chi(t)$ and $\chi_\ell(t)$ for the characteristic polynomials of S over R and $\overline{S}$ over $\mathbf{k}[x, y]$, respectively.

As in Lemma 2.1 we can find a graded $\chi_\ell(t)$-Roby module

$$\phi_\ell : \overline{S} \oplus \mathbf{k}[x, y] \cdot T \to \operatorname{End}(W) \otimes \mathbf{k}[x, y]$$

such that C_{ϕ_ℓ} is a morphism. Recall that ϕ_ℓ must have the property that

$$\phi_\ell(\alpha_1\gamma_1 + \ldots + \alpha_d\gamma_d + \tau T)^d = \chi_\ell(\alpha_1\gamma_1 + \ldots + \alpha_d\gamma_d + \tau t) \cdot \mathrm{id}_W$$

where $\alpha_\bullet, \tau \in \mathbf{k}[x, y]$. Since $\mathbf{k}[x, y]$ is naturally a subring of R, we may define $\phi_0 := \phi_\ell \otimes_{\mathbf{k}[x,y]} R$. Write $\chi_0(t)$ for $\chi_\ell(t)$ viewed as an element of $\operatorname{Sym}^\bullet_R(S^\vee)[t]$. Then ϕ_0 is a graded $\chi_0(t)$-Roby module.

Now we note that $\chi(t) - \chi_0(t) \in (z_\bullet) \operatorname{Sym}^\bullet_R(S^\vee)[t]$. So let us write

$$\chi(t) = \chi_0(t) + \sum_{i=0}^{d-1} \sum_{j=1}^{n_i} t^i (c_{i,j,1}\Gamma_{k(i,j,1)})(c_{i,j,2}\Gamma_{k(i,j,2)}) \cdots (c_{i,j,d-i}\Gamma_{k(i,j,d-i)})$$

where $\Gamma_1, \ldots, \Gamma_d$ are the variables dual to the basis $\gamma_1, \ldots, \gamma_d$ and $c_{i,j,s} \in R$ has degree equal to $\deg(\gamma_{k(i,j,s)})$. Now put $m_{i,j} = t^i (c_{i,j,1}\Gamma_{k(i,j,1)}) \ldots (c_{i,j,d-i}\Gamma_{k(i,j,d-i)})$.

We recall the construction of Example 1.3. Define a map $\phi_{m_{i,j}} : S \oplus R\{T\} \to \operatorname{End}(R \otimes \mathbf{k}^d)$ by

$$\phi_{m_{i,j}}(\gamma_p)(\epsilon_r) = c_{i,j,r-i}\delta^p_{k(i,j,r-i)}\epsilon_{r+1}, \quad (i < r \leq d), \quad \phi_{m_{i,j}}(T)(\epsilon_r) = \begin{cases} \epsilon_{r+1} & 1 \leq r \leq i, \\ 0 & i < r \leq d \end{cases}$$

where $\epsilon_\bullet$ are the standard basis vectors of $\mathbf{k}^d$ and addition in the subscripts are modulo d.

Now consider $\phi = \phi_0 \widehat{\otimes}_\xi \phi_{m_{0,1}} \widehat{\otimes}_\xi \cdots \widehat{\otimes}_\xi \phi_{m_{d,n_d}}$ which is a $\chi_{\mathcal{A}}-$Roby action on $\widetilde{W} = (\mathbf{k}^d)^{\otimes n_0 + \cdots + n_{d-1}} \otimes W \otimes R$. By Proposition 1.15, this is a χ-Roby module. For each i, j at least one of the $c_{i,j,r}$ must be in the ideal $(z_\bullet)$. After possibly reindexing, we may assume that $c_{i,j,d-i} \in (z_\bullet)$.

Consider the filtration $F^i\mathbf{k}^d = \mathbf{k}\{\epsilon_i, \ldots, \epsilon_d\}$ on each "monomial" Roby module. Then upon restriction to ℓ, the Roby-action of $\overline{S} \oplus \mathbf{k}[x, y]\{T\}$ on $\mathbf{k}^{\times d} \otimes \mathbf{k}[x, y]$ via $\phi_{m_{i,j}}$ preserves $F^\bullet$. Moreover, if we put $F^{d+1} = 0$, we have that for each i,

$$(\overline{S} \oplus \mathbf{k}[x, y]\{T\}) \cdot F^i \subset F^{i+1}$$

Let us equip $\widetilde{W}$ with the filtration $\widehat{F}^\bullet$ which is the tensor product of the filtrations above on its monomial factors and the trivial filtration on $W \otimes R$. Then the action of $\overline{S} \oplus \mathbf{k}[x, y]\{T\}$ preserves $\widehat{F}^\bullet$. The formation of the $\mathbb{Z}/d\mathbb{Z}-$graded tensor product is bi-functorial on Roby modules. Since each $m_{i,j}$ vanishes in $\mathbf{k}[x, y, t]$, we see that the minimal subquotients of $\widehat{F}^\bullet$ are simply the $\mathbb{Z}/d\mathbb{Z}-$graded tensor product of ϕ_ℓ with a number of copies of the rank-one 0-Roby module correpsonding to the zero map $\overline{S} \oplus \mathbf{k}[x, y]\{T\} \to \mathrm{End}(\mathbf{k}[x, y])$. Therefore the minimal subquotients are isomorphic to ϕ_ℓ.

Finally, the formation of C_ϕ is functorial. So the action of $\overline{S}$ via C_ϕ preserves the tensor product filtration. Since the associated graded parts are isomorphic to ϕ_ℓ and C_{ϕ_ℓ} is a morphism, we find that $C_\phi|_\ell$ is a filtered pseudo-morphism. □

Lemma 2.6. *Let $\mathcal{A}$ be an ACM sheaf of algebras on $\mathbb{P}^N$ and $\ell \subset \mathbb{P}^N$ a line. Let $\widehat{\mathcal{A}}$ be the formal completion of $\mathcal{A}$ along ℓ and let $\widehat{\mathcal{E}}$ be a coherent sheaf of $\widehat{\mathcal{A}}$-modules. Then there is a coherent sheaf $\mathcal{E}$ of $\mathcal{A}$-modules whose completion is isomorphic to $\widehat{\mathcal{E}}$.*

Proof. Let $\mathcal{I}$ be the ideal defining ℓ. Consider the sheaf of algebras $\mathcal{S} = \oplus_{m\geq 0}\mathcal{I}^m/\mathcal{I}^{m+1}$ and the coherent graded $\mathcal{S}$ module $\mathcal{F} = \oplus_{m\geq 0}\mathcal{I}^m\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}}$. Note that $\mathcal{S}(1)$ is ample on $\mathrm{Spec}(\mathcal{S})$, where $\mathcal{S}(1)$ is the pullback of $\mathcal{O}_\ell(1)$ under the natural map $\mathrm{Spec}(\mathcal{S}) \to \ell$. It follows that for some $n_0 \gg 0$, $\mathrm{H}^1(\mathcal{F}(n_0)) = 0$. Observe that

$$\mathrm{H}^1(\mathrm{Spec}(\mathcal{S}), \mathcal{F}(n_0)) = \oplus_m \mathrm{H}^1(\ell, (\mathcal{I}^m\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}})(n_0)),$$

since $\mathrm{Spec}(\mathcal{S}) \to \ell$ is affine. Therefore, for each m,

$$\mathrm{H}^1((\mathcal{I}^m\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}})(n_0)) = 0.$$

It follows that the maps

$$\mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}^{m+1}\widehat{\mathcal{E}})(n_0)) \to \mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}^m\widehat{\mathcal{E}})(n_0))$$

are surjective. Therefore the map $\mathrm{H}^0(\widehat{\mathcal{E}}(n_0)) \to \mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}\widehat{\mathcal{E}})(n_0))$ is surjective. If $V \subset \mathrm{H}^0((\widehat{\mathcal{E}}/\mathcal{I}\widehat{\mathcal{E}})(n_0))$ is a finite dimensional space of sections such that

$$V \otimes \mathcal{A}(-n_0) \to \widehat{\mathcal{E}}/\mathcal{I}\widehat{\mathcal{E}}$$

is surjective and $V' \subset \mathrm{H}^0(\widehat{\mathcal{E}}(n_0))$ is a lift then

$$V' \otimes \widehat{\mathcal{A}}(-n_0) \to \widehat{\mathcal{E}}$$

is also surjective. Indeed, the support of the cokernel is empty. Iterating this arugment we obtain a presentation

$$W \otimes \widehat{\mathcal{A}}(-n_1) \xrightarrow{\hat{\alpha}} V' \otimes \widehat{\mathcal{A}}(-n_0) \to \widehat{\mathcal{E}} \to 0.$$

Now consider the map $\mathrm{H}^0(\mathcal{A}(k)) \to \mathrm{H}^0(\widehat{\mathcal{A}}(k))$. We wish to show that it is surjective. Since $\mathcal{A}$ is dissocié, it suffices to show that the maps $\mathrm{H}^0(\mathcal{O}(k)) \to \mathrm{H}^0(\widehat{\mathcal{O}}(k))$ are surjective for all $k \gg 0$. If $m > k$ then $\mathrm{H}^0(\mathcal{O}(k)) \to \mathrm{H}^0((\mathcal{O}/\mathcal{I}^m)(k))$ is an isomorphism. Hence $\mathrm{H}^0(\mathcal{A}(k)) \to \mathrm{H}^0(\widehat{\mathcal{A}}(k))$ is an isomorphism. Therefore there is a morphism

$$\alpha : W \otimes \mathcal{A}(-n_1) \to V' \otimes \mathcal{A}(-n_0)$$

whose completion is $\hat{\alpha}$. Thus we may take $\mathcal{E} = \mathrm{coker}(\alpha)$. □

The next result completes the proof of Theorem A.

Theorem 2.7. *Under the hypothesis of Proposition 2.5, there exists a 1-dimensional linear section $C \subseteq X$ and a reflexive sheaf $\mathcal{E}$ on X such that $\mathcal{E}|_C$ is Ulrich.*

Proof. Let $\pi : X \to \mathbb{P}^n$ and V_1, V_2 be as in Lemma 2.4 and let $\ell \subset V_1 \cup V_2$ be a line. Next, let $(W, F^\bullet)$ be a filtered vector space and $\phi : \mathcal{A} = \pi_* \mathcal{O}_X \to \operatorname{End}(W \otimes \mathcal{O})$ be as in Proposition 2.5 with respect to the line ℓ. Let $z_i \in \mathcal{A}(V_i)$ be an algebra generator for $\mathcal{A}$ over V_i. Consider the algebra map ϕ_i defined by the diagram

$$\mathcal{A}_i = \mathcal{A}_{V_i} \xleftarrow{\cong} \mathcal{O}_{V_i}[z_i]/(p_i(z_i)) \longrightarrow \operatorname{End}(W \otimes \mathcal{O}_{V_i})$$

where the second map is the unique algebra homomorphism which sends z_i to $\phi(z_i)$. Write $\mathcal{E}_i$ for $W \otimes \mathcal{O}_{V_i}$ with the $\mathcal{A}_{V_i}$-module structure coming from ϕ_i. Note that $\mathcal{E}_i$ is maximal Cohen-Macaulay over V_i and therefore locally free on $V_i \setminus p(\operatorname{sing}(X))$; in particular, $\mathcal{E}_i$ is locally free in a neighborhood of $\ell \cap V_i$. For $i, j = 1, 2,\ i \neq j$, we denote the restriction of the $\mathcal{A}_{V_i}-$module $\mathcal{E}_i$ to $V_{ij} = V_i \cap V_j$ by $\mathcal{E}_{ij}$.

Let $F^\bullet$ be the filtration on W. By assumption, the pseudomorphism $\phi : \mathcal{A}|_\ell \to \operatorname{End}(W \otimes \mathcal{O}_\ell)$ induces an $\mathcal{O}_\ell-$algebra morphism $\mathcal{A}|_\ell \to \prod \operatorname{End}(F^{i+1}W/F^iW \otimes \mathcal{O}_\ell)$. Since X_ℓ is smooth, the $\mathcal{A}|_\ell$-module structure on $F^{i+1}W/F^iW \otimes \mathcal{O}_\ell$ is locally free. Now, $V_1 \cap V_2 \cap \ell$ is affine. This means that the filtration $F^\bullet \mathcal{E}_{ij}$ has projective subquotients. So there is an isomorphism $\operatorname{gr}_{F^\bullet} \mathcal{E}_{ij} \to \mathcal{E}_{ij}$ which is compatible with the filtration when $\operatorname{gr}_{F^\bullet} \mathcal{E}_{ij}$ is filtered by $F^k = \oplus_{k' \leq k} F^{k'} \mathcal{E}_{ij} / F^{k'-1} \mathcal{E}_{ij}$ and which induces the identity on subquotients. Using these isomorphisms we produce a filtered $\mathcal{A}_{12}$-module isomorphism

$$\psi_\ell : \mathcal{E}_{12}|_\ell \to \mathcal{E}_{21}|_\ell$$

which induces the same isomorphism on associated graded modules as the identification $\mathcal{E}_{12} = W \otimes \mathcal{O}_{\ell \cap V_{12}} = \mathcal{E}_{21}$. Let $\mathcal{F}$ be the vector bundle on ℓ obtained by gluing $\mathcal{E}_1|_\ell$ to $\mathcal{E}_2|_\ell$ along ψ_ℓ. Now since ψ_ℓ is filtered, $\mathcal{F}$ is filtered. By construction, the subquotients of $\mathcal{F}$ for this filtration are the same as the subquotients for $W \otimes \mathcal{O}_\ell$. Hence the associated graded of $\mathcal{F}$ is trivial. It follows that $\mathcal{F}$ is itself a trivial vector bundle.

Let $\widehat{U}$ be the formal neighborhood of ℓ in $\mathbb{P}^n$. Let $j : \widehat{U} \to \mathbb{P}^n$ and put $\widehat{V}_i = j^{-1}(V_i)$ and $\widehat{\mathcal{A}} = j^* \mathcal{A}$. Write $\widehat{\mathcal{E}}_i = j^* \mathcal{E}_i$. Since $\mathcal{E}_i$ is a locally free $\mathcal{A}_i$ module on a neighborhood of $\ell \cap V_i$ we see that $\widehat{\mathcal{E}}_i$ is a locally free $\widehat{\mathcal{A}}$-module. Since $\mathbb{P}^n$ is separated, $V_{12} = V_1 \cap V_2$ is affine. Hence $\widehat{V}_{12} = \widehat{V}_1 \cap \widehat{V}_2$ is an affine formal scheme. Hence the $\widehat{\mathcal{E}}_i$ are projective $\widehat{\mathcal{A}}_{V_i}$-modules. Therefore the isomorphism ψ_ℓ lifts to an isomorphism

$$\psi : \widehat{\mathcal{E}}_{12} \to \widehat{\mathcal{E}}_{21}$$

of $\widehat{\mathcal{A}}_{12}$-modules. The isomorphism ψ gives gluing data for gluing $\widehat{\mathcal{E}}_1$ to $\widehat{\mathcal{E}}_2$. Let $\widehat{\mathcal{E}}$ be $\widehat{\mathcal{A}}$-module obtained by gluing $\widehat{\mathcal{E}}_1$ to $\widehat{\mathcal{E}}_2$ along ψ. By Lemma 2.6, there is a sheaf $\mathcal{E}$ of $\mathcal{A}$-modules whose restriction to $\widehat{U}$ is $\widehat{\mathcal{E}}$. Since $\widehat{\mathcal{E}}$ is locally free, $\mathcal{E}$ is locally free in a neighborhood of ℓ. So replacing $\mathcal{E}^{\vee\vee}$ is also isomorphic to $\widehat{\mathcal{E}}$ on $\widehat{U}$. Hence $\mathcal{E}^{\vee\vee}$ is the desired sheaf. □

3 Generalities on $\delta-$Ulrich Sheaves

In this section, $X \subseteq \mathbb{P}^N$ is a normal ACM variety of degree d and dimension $n \geq 2$.

Lemma 3.1. *Suppose that $\mathcal{E}$ is a locally CM sheaf on X whose restriction to a linear section Y of dimension at least 2 is Ulrich. Then $\mathcal{E}$ is Ulrich.*

Proof. Let $\pi : X \to \mathbb{P}^n$ be a finite linear projection. Since $\mathcal{E}$ is a locally CM sheaf on X, the direct image $\pi_* \mathcal{E}$ is a locally CM sheaf on a smooth variety, and is therefore locally free. Replacing $\mathcal{E}$ by $\pi_* \mathcal{E}$ if necessary, we can assume without loss of generality that $\mathcal{E}$ is locally free and $X = \mathbb{P}^n$. We will show that $\mathcal{E}$ is trivial.

By induction on dimension we may assume that $\dim(Y) = n-1$ so that Y is a hyperplane. Our hypothesis on $\mathcal{E}$ amounts to $\mathcal{E}|_Y$ being a trivial bundle. To show that $\mathcal{E}$ is trivial, it is enough to check that $h^0(\mathcal{E}) = \mathrm{rk}(\mathcal{E})$. (See Proposition 3.2.) We will do this by showing that restriction map $H^0(\mathcal{E}) \to H^0(\mathcal{E}|_Y)$ is surjective, since $h^0(\mathcal{E}|_Y) = \mathrm{rk}(\mathcal{E})$.

For each positive integer j, let jY the $(j-1)-$st order thickening of Y. We claim that for all $m \geq 1$, the restriction map $H^0(\mathcal{E}|_{(m+1)Y}) \to H^0(\mathcal{E}|_{mY})$ is an isomorphism. Grant this for the time being. If we fix $m_0 \gg 0$ for which $H^1(\mathcal{E}(-m_0)) = 0$, then the restriction map $H^0(\mathcal{E}) \to H^0(\mathcal{E}|_{m_0 Y})$ is surjective, and the claim yields the desired surjectivity of the map $H^0(\mathcal{E}) \to H^0(\mathcal{E}|_Y)$.

Turning to the proof of this claim, an obvious snake lemma argument gives the following exact sequence for each $m \geq 1$:

$$0 \to \mathcal{E}|_Y(-m) \to \mathcal{E}|_{(m+1)Y} \to \mathcal{E}|_{mY} \to 0.$$

Since $\mathcal{E}|_Y$ is trivial and $\dim(Y) > 1$, $h^0(\mathcal{E}|_Y(-m)) = h^1(\mathcal{E}|_Y(-m)) = 0$, so the map $H^0(\mathcal{E}|_{(m+1)Y}) \to H^0(\mathcal{E}|_{mY})$ is an isomorphism.

□

If $\mathcal{E}$ is a $\delta-$Ulrich sheaf on X then for a general 1-dimensional linear section $Y \subset X$, $\mathcal{E}|_Y$ is Ulrich. Indeed, if we consider a finite linear projection $\pi : X \to \mathbb{P}^n$ then $\pi_*\mathcal{E}$ is a reflexive sheaf whose restriction to a given line is trivial. Since trivial sheaves on $\mathbb{P}^1$ are rigid, the restriction of $\pi_*\mathcal{E}$ to nearby lines is also trivial. We also point out that if X_H is a general hyperplane section of X then $\mathcal{E}|_{X_H}$ is δ-Ulrich on X_H.

Proposition 3.2. *Let $\mathcal{E}$ be a $\delta-$Ulrich sheaf of rank r on X. Then the following are equivalent.*

(i) *$\mathcal{E}$ is Ulrich.*

(ii) *$h^0(\mathcal{E}) = dr$.*

Proof. The implication $(i) \Rightarrow (ii)$ is clear, so we will focus on proving $(ii) \Rightarrow (i)$. Assume $h^0(\mathcal{E}) = dr$. If $\pi : X \to \mathbb{P}^n$ is a finite linear projection, then $h^0(\mathcal{E}) = h^0(\pi_*\mathcal{E})$, and $\mathcal{E}$ is Ulrich if and only if $\pi_*\mathcal{E}$ is Ulrich with respect to $\mathcal{O}_{\mathbb{P}^n}(1)$, i.e. a trivial vector bundle. We may then assume without loss of generality that $(X, \mathcal{O}(1)) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ (in particular, $d = 1$). In this case the $\delta-$Ulrich condition on $\mathcal{E}$ is equivalent to $\mathcal{E}|_\ell \cong \mathcal{O}_\ell^r$ for some line $\ell \subseteq \mathbb{P}^n$, and the Ulrich condition on $\mathcal{E}$ is equivalent to $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n}^r$.

If $h^0(\mathcal{E}) = r$, then since $h^0(\mathcal{E} \otimes \mathcal{I}_\ell) = 0$ we find that the evaluation map $ev : \mathrm{H}^0(\mathcal{E}) \otimes \mathcal{O} \to \mathcal{E}$ restricts to the evaluation map $\mathrm{H}^0(\mathcal{E}|_\ell) \otimes \mathcal{O}_\ell \to \mathcal{E}|_\ell$, which is an isomorphism. If $\mathcal{F}$ is the cokernel of ev, its support cannot intersect ℓ and therefore has codimension at least two. We then have that $\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}^r) \cong (H^{n-1}(\mathcal{F}(-n-1))^*)^r \cong 0$. Since $\mathcal{E}$ is reflexive, it is torsion-free, so we must have $\mathcal{F} = 0$, i.e. that ev is an isomorphism. □

We shall now consider stability properties of $\delta-$Ulrich bundles.

Lemma 3.3. *Let $\mathcal{E}$ be a δ-Ulrich sheaf on X. Then $\mathcal{E}$ is $\mu-$semistable, and $\omega_X \otimes \mathcal{E}^\vee(n+1)$ is also δ-Ulrich.*

Proof. Let $\mathcal{F}$ be a torsion-free quotient of $\mathcal{E}$, and let $Y \subset X$ be a smooth 1-dimensional linear section such that $\mathcal{E}|_Y$ is Ulrich; we may also assume Y avoids the singular loci of $\mathcal{E}$ and $\mathcal{F}$. Then $\mathcal{F}|_Y$ is a torsion-free quotient of the semistable bundle $\mathcal{E}|_Y$; consequently $\mu(\mathcal{E}) = \mu(\mathcal{E}|_Y) \leq \mu(\mathcal{F}_Y) = \mu(\mathcal{F})$.

The second part of the statement follows from the adjunction formula and the fact that if C is a curve embedded in projective space by $\mathcal{O}_C(1)$ and $\mathcal{E}'$ is an Ulrich bundle on C, then $\omega_C \otimes \mathcal{E}'^\vee(2)$ is also Ulrich. □

Lemma 3.4. *If $\mathcal{E}$ is a δ-Ulrich sheaf on X which is strictly $\mu-$semistable, then there exists a $\mu-$stable subsheaf $\mathcal{E}' \subset \mathcal{E}$ which is $\delta-$Ulrich.*

Proof. Let $\mathcal{E}' \subset \mathcal{E}$ be the maximal destabilizing subsheaf of $\mathcal{E}$. We will show that $\mathcal{E}'$ is $\delta-$Ulrich. Let $Y \subseteq X$ be a general 1-dimensional linear section of $\mathcal{E}$ which avoids the singular loci of $\mathcal{E}'$ and $\mathcal{E}$. Then $\mathcal{E}'|_Y(-1)$ is a subsheaf of $\mathcal{E}|_Y(-1)$. Since the latter is an Ulrich sheaf, we have that $h^0(\mathcal{E}|_Y(-1)) = 0$, and it follows that $h^0(\mathcal{E}'|_Y(-1)) = 0$ as well. The slope of $\mathcal{E}'|_Y(-1)$ is equal to that of $\mathcal{E}|_Y(-1)$, so Riemann-Roch implies that $h^0(\mathcal{E}'|_Y(-1)) = h^1(\mathcal{E}'|_Y(-1)) = 0$; therefore $\mathcal{E}'|_Y$ is Ulrich. ☐

Lemma 3.5. *Let $\mathcal{E}$ be a δ-Ulrich sheaf on X. Then for all $k \geq 1$, we have $h^0(\mathcal{E}(-k)) = 0$.*

Proof. We proceed by induction on $\dim(X)$. Let $X_H \subset X$ be a general hyperplane section. Then for each $k \geq 1$ we have the exact sequence

$$0 \to \mathcal{E}(-k-1) \to \mathcal{E}(-k) \to \mathcal{E}|_{X_H}(-k) \to 0$$

Since negative twists of an Ulrich sheaf have no global sections, our inductive hypothesis implies $h^0(\mathcal{E}_{X_H}(-k)) = 0$; it follows that $h^0(\mathcal{E}(-k)) = h^0(\mathcal{E}(-1))$ for all $k \geq 1$. We need only exhibit some $k' \geq 1$ such that $h^0(\mathcal{E}(-k')) = 0$. Since $\mathcal{E}$ and all its twists are μ-semistable by Lemma 3.3, any positive $k' > \mu(\mathcal{E})$ will do. ☐

Remark 3.6. We exhibit for each $n \geq 2$ a smooth ACM variety of dimension n admitting $\delta-$Ulrich sheaves which are not Ulrich. Consider the Segre variety $X := \mathbb{P}^1 \times \mathbb{P}^{n-1} \subseteq \mathbb{P}^{2n-1}$, and let H be the hyperplane class of X. Recall that X is cut out in $\mathbb{P}^{2n+1}$ by the maximal minors of the generic $2 \times n$ matrix of linear forms. It follows from Proposition 2.8 of [BHU87] that the degeneracy locus $D \subseteq X$ of the first row of this matrix is a divisor whose associated line bundle $\mathcal{O}_X(D)$ is an Ulrich line bundle on X. The general 1-dimensional linear section $X' \subset X$ is a rational normal curve of degree n, so if $\mathcal{L} \in \mathrm{Pic}(X)$ satisfies $H^{n-1} \cdot \mathcal{L} = 0$, the restriction $\mathcal{L}|_{X'}$ is the trivial bundle; in particular $\mathcal{L}(D)$ is $\delta-$Ulrich. Since the set $(H^{n-1})^{\perp}$ of all such $\mathcal{L}$ is a corank-1 subgroup of $\mathrm{Pic}(X)$, we can choose $\mathcal{L} \in (H^{n-1})^{\perp}$ such that $\mathcal{L}(D)$ lies outside the effective cone of X, e.g. satisfies $H^0(\mathcal{L}(D)) = 0$. In this case $\mathcal{L}(D)$ is not Ulrich.

4 The surface case

Throughout this section we consider a normal surface X with a very ample line bundle $\mathcal{O}_X(1)$. We assume that X has a δ-Ulrich sheaf $\mathcal{E}$, but not necessarily that X is ACM.

4.1 Relation to Instanton Sheaves

Proposition 4.1. *Let $\mathcal{E}$ be a $\delta-$Ulrich sheaf of rank r on X, and let $\pi : X \to \mathbb{P}^2$ be a finite linear projection. Then $\pi_*\mathcal{E}$ is $\mu-$semistable, and it is an instanton sheaf on $\mathbb{P}^2$, i.e. the cohomology of a monad of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus m} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus rd+2m} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus m} \to 0 \qquad (1)$$

where $d = \deg(X)$ and $m = h^1(\mathcal{E}(-1))$.

Proof. If $\mathcal{E}$ is $\delta-$Ulrich, then $\pi_*\mathcal{E}$ is a reflexive, and thus locally free, sheaf on $\mathbb{P}^2$. Since the restriction of $\pi_*\mathcal{E}$ to a general line is trivial, $\pi_*\mathcal{E}$ is $\mu-$semistable of degree 0, and given that $h^1(\mathcal{E}(-1)) = h^1(\pi_*\mathcal{E}(-1))$, Theorem 17 of [Jar06] implies our result. ☐

The following statement can be obtained from a short elementary argument, but it seems appropriately stated as a consequence of Proposition 4.1.

Corollary 4.2. *A $\delta-$Ulrich sheaf $\mathcal{E}$ on X is Ulrich if and only if $H^1(\mathcal{E}(-1)) = 0$.* □

At this point it is natural to ask if, given a $\delta-$Ulrich sheaf $\mathcal{E}$ on X, there is a $\delta-$Ulrich sheaf $\mathcal{E}'$ on X with $h^1(\mathcal{E}'(-1)) < h^1(\mathcal{E}(-1))$; an affirmative answer combined with Theorem A would imply that every normal ACM surface admits an Ulrich sheaf. The next result shows that it is enough to consider stable $\delta-$Ulrich bundles.

Lemma 4.3. *Let X be a smooth ACM surface, and let $\mathcal{E}$ be a δ-Ulrich sheaf on X which is strictly μ-semistable with $h^1(\mathcal{E}(-1)) = m$. Then X admits a locally free δ-Ulrich sheaf $\mathcal{E}'$ with $\mathrm{rk}(\mathcal{E}') < \mathrm{rk}(\mathcal{E})$ and $h^1(\mathcal{E}'(-1)) \leq \frac{m}{2}$.*

Proof. A μ-Jordan-Hölder filtration of $\mathcal{E}$ yields an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{E}(-1) \to \mathcal{G}(-1) \to 0$$

where $\mathcal{F}$ and $\mathcal{G}$ are both δ-Ulrich sheaves and $\mathcal{F}$ is both $\mu-$stable and locally free. Since $h^2(\mathcal{F}(-1)) = h^0(\omega_X \otimes \mathcal{F}^\vee(1))$, and $\omega_X \otimes \mathcal{F}^\vee(3)$ is δ-Ulrich by Lemma 3.3, we have from Lemma 3.5 that $h^2(\mathcal{F}(-1)) = 0$. Another application of this Lemma implies that $h^0(\mathcal{G}(-1)) = 0$. We may then conclude that $\min\{h^1(\mathcal{F}(-1)), h^1(\mathcal{G}(-1))\} \leq \frac{m}{2}$, and the result follows by taking the reflexive hull of $\mathcal{G}$ if necessary. □

Remark 4.4. Suppose that $X \subseteq \mathbb{P}^N$ is an ACM variety of dimension n and let $\pi : X \to \mathbb{P}^n$ be a finite linear projection. Jardim [Jar06] defines a notion of instanton sheaves on $\mathbb{P}^n$ for any $n > 1$; according to his definition, a $\mu-$semistable reflexive instanton sheaf is $\delta-$Ulrich. However, only in the case $n = 2$ is the direct image $\pi_*\mathcal{E}$ of a $\delta-$Ulrich sheaf on X clearly an instanton sheaf. For $n > 2$ an instanton sheaf must satisfy additional cohomology-vanishing which does not follow from having trivial restriction to a line. Our construction does not appear to allow for any control over the cohomology of $\delta-$Ulrich sheaves.

Remark 4.5. Instanton sheaves can be used to show that a smooth projective threefold $X \subseteq \mathbb{P}^N$ admitting an Ulrich sheaf (e.g. a smooth complete intersection of $N-3$ hypersurfaces, by [BHU91]) admits a $\delta-$Ulrich sheaf which is not locally free. Let $\pi : X \to \mathbb{P}^3$ be a general linear projection, and let $z \in \mathbb{P}^3$ be a point not contained in the branch divisor of π. By Example 5 in [Jar06] there exists a rank-3 $\mu-$semistable reflexive instanton (and thus $\delta-$Ulrich) sheaf $\mathcal{G}$ on $\mathbb{P}^3$ whose singular locus is exactly z. If $\mathcal{F}$ is an Ulrich sheaf on X, then $\mathcal{F}$ is locally free by the smoothness of X, and $\mathcal{E} := \mathcal{F} \otimes \pi^*\mathcal{G}$ is a reflexive sheaf on X whose singular locus is exactly $\pi^{-1}(z)$. Since $\mathcal{G}$ is locally free away from z, the sheaves $\pi_*\mathcal{E}$ and $\pi_*\mathcal{F} \otimes \mathcal{G}$ are isomorphic away from z. Consequently there exists a line $\ell \subseteq \mathbb{P}^3$ not containing z such that the restricton of $\pi_*\mathcal{E}$ to ℓ is a trivial bundle. It follows that $\mathcal{E}$ is $\delta-$Ulrich.

4.2 Proof of Theorem B

For the next two Lemmas, we consider a δ-Ulrich sheaf $\mathcal{E}$ on $\mathbb{P}^2$ for the canonical polarization $\mathcal{O}_{\mathbb{P}^2}(1)$. In general, it is difficult to understand how an abstract δ-Ulrich sheaf will restrict to a curve in $\mathbb{P}^2$. However, we can say something when the curve is a general smooth conic.

Lemma 4.6. *Let $C \subset \mathbb{P}^2$ be a general smooth conic. Then $\mathcal{E}|_C$ is trivial.*

Proof. Any smooth conic is isomorphic to $\mathbb{P}^1$, so it is enough to show that $\mathcal{E}|_C$ is of degree 0 and semistable when C is a general element of $|\mathcal{O}_{\mathbb{P}^2}(2)|$. Since the restriction of $\mathcal{E}$ to a general line is a trivial bundle, it follows that $\det(\mathcal{E})$ is trivial. Consequently the restriction of $\mathcal{E}$ to any plane curve has degree 0. We now turn to semistability. Consider the universal plane conic

$$\mathcal{C} := \{(p, C) \in \mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)| : p \in C\}$$

with its associated projections $p_1 : \mathcal{C} \to \mathbb{P}^2, p_2 : \mathcal{C} \to |\mathcal{O}_{\mathbb{P}^2}(2)|$. Our goal amounts to showing that the restriction of $p_1^*\mathcal{E}$ to the general fiber of p_2 is semistable. Given that this is an open condition on the fibers of p_2 (e.g. Proposition 2.3.1 in [HL10]) it is enough to check the semistability of $\mathcal{E}|_{C_0}$ when $C_0 = L \cup L'$ for distinct lines $L, L' \subseteq \mathbb{P}^2$ satisfying the property that $\mathcal{E}|_L$ and $\mathcal{E}|_{L'}$ are trivial. If we twist the Mayer-Vietoris sequence

$$0 \to \mathcal{O}_{C_0} \to \mathcal{O}_L \oplus \mathcal{O}_{L'} \to \mathcal{O}_{L\cap L'} \to 0$$

by $\mathcal{E}$ and take cohomology, we see that the induced difference map $H^0(\mathcal{E}|_L) \oplus H^0(\mathcal{E}|_{L'}) \to H^0(\mathcal{E}|_{L\cap L'})$ is surjective. Therefore $\mathcal{E}|_{C_0}$ is locally free of rank $\mathrm{rk}(\mathcal{E})$ with $\mathrm{rk}(\mathcal{E})$ global sections, i.e. $\mathcal{E}|_{C_0} \cong \mathcal{O}_{C_0}^{\oplus \mathrm{rk}(\mathcal{E})}$. In particular, $\mathcal{E}|_{C_0}$ is semistable. □

Lemma 4.7. *Let $\mathcal{F}$ be an $\mathcal{O}(2)$-Ulrich sheaf on $\mathbb{P}^2$. Then $\mathcal{E} \otimes \mathcal{F}$ is δ-Ulrich for $\mathcal{O}(2)$ and we have*

$$\chi(\mathcal{E} \otimes \mathcal{F}) = \mathrm{rk}(\mathcal{F})\,(\chi(\mathcal{E}) + 3\mathrm{rk}(\mathcal{E}))\,.$$

Proof. Since the restriction of $\mathcal{E}$ and $\mathcal{F}$ to a general conic are trivial and Ulrich, respectively, and Ulrich sheaves are stable under taking direct sums, we see that the restriction of $\mathcal{E} \otimes \mathcal{F}$ to a general conic is Ulrich. Hence $\mathcal{E} \otimes \mathcal{F}$ is $\delta-$Ulrich for $\mathcal{O}(2)$.

Since $\mathcal{E}$ is $\delta-$Ulrich for $\mathcal{O}_{\mathbb{P}^2}(1)$, Proposition 4.1 implies that it is the cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^m \to \mathcal{O}_{\mathbb{P}^2}^{\mathrm{rk}(\mathcal{E})+2m} \to \mathcal{O}_{\mathbb{P}^2}(1)^m \to 0$$

where $m = h^1(\mathcal{E}(-1))$. Twisting by $\mathcal{F}$, we have that if $\ell \subseteq \mathbb{P}^2$ is a line, then

$$\begin{aligned}\chi(\mathcal{E} \otimes \mathcal{F}) &= (\mathrm{rk}(\mathcal{E}) + 2m) \cdot \chi(\mathcal{F}) - m \cdot (\chi(\mathcal{F}(-1)) + \chi(\mathcal{F}(1)))\\ &= \mathrm{rk}(\mathcal{E}) \cdot \chi(\mathcal{F}) + m \cdot ((\chi(\mathcal{F}) - \chi(\mathcal{F}(-1))) - (\chi(\mathcal{F}(1)) - \chi(\mathcal{F})))\\ &= \mathrm{rk}(\mathcal{E}) \cdot \chi(\mathcal{F}) + m \cdot (\chi(\mathcal{F}|_\ell) - \chi(\mathcal{F}(1)|_\ell))\\ &= \mathrm{rk}(\mathcal{E}) \cdot \chi(\mathcal{F}) - m \cdot \mathrm{rk}(\mathcal{F})\end{aligned}$$

We have from Riemann-Roch that $\chi(\mathcal{E}) = \mathrm{ch}_2(\mathcal{E}) + \mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}) - m$; also, the fact that $\mathcal{F}$ is Ulrich with respect to $\mathcal{O}(2)$ implies that $\chi(\mathcal{F}) = 4\mathrm{rk}(\mathcal{F})$. Summarizing, we have that

$$\chi(\mathcal{E} \otimes \mathcal{F}) = 4\mathrm{rk}(\mathcal{E}) \cdot \mathrm{rk}(\mathcal{F}) + (\chi(\mathcal{E}) - \mathrm{rk}(\mathcal{E})) \cdot \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F})\,(\chi(\mathcal{E}) + 3\mathrm{rk}(\mathcal{E}))\,.$$

□

One way to explain the previous Lemma is that Ulrich sheaves on $\mathbb{P}^2$ for $\mathcal{O}(2)$ are slightly positive. (The main example of an $\mathcal{O}(2)$-Ulrich sheaf is the tangent bundle $T\mathbb{P}^2$.) So tensoring with such a sheaf should enlarge the space of sections while decreasing the the higher cohomology. The Lemma makes this intuition precise and the next Theorem uses this idea to produce δ-Ulrich sheaves with sections (after changing the polarization).

Theorem 4.8. *Assume that $(X, \mathcal{O}_X(1))$ admits a δ-Ulrich sheaf. Then there exists a sequence $\mathcal{E}_m$ of sheaves on X such that $\mathcal{E}_m$ is δ-Ulrich for $\mathcal{O}_X(2^m)$ and*

$$\lim_{m\to\infty} \alpha(\mathcal{E}_m) = 1.$$

In particular, for $m \gg 0$, $h^0(\mathcal{E}_m) > 0$.

Proof. We will construct the sequence $\mathcal{E}_m$ inductively as follows. Put $\mathcal{E}_0 = \mathcal{E}$ and fix an $\mathcal{O}(2)$-Ulrich sheaf $\mathcal{F}$ on $\mathbb{P}^2$. Now, assume we have constructed $\mathcal{E}_0, \cdots, \mathcal{E}_m$ such that $\mathcal{E}_i$ is δ-Ulrich with respect to $\mathcal{O}_X(2^i)$. To construct $\mathcal{E}_{m+1}$ we consider the embedding $X \to \mathbb{P}^N$ determined by $\mathcal{O}_X(2^m)$. Let $\pi : X \to \mathbb{P}^2$ be a finite map obtained as the compostion of i with a general linear projection $\mathbb{P}^N \dashrightarrow \mathbb{P}^2$. Define $\mathcal{E}_{m+1} = \mathcal{E}_m \otimes \pi^*\mathcal{F}$. By Lemma 4.7, $\pi_*(\mathcal{E}_m \otimes \pi^*\mathcal{F}) = \pi_*(\mathcal{E}_m) \otimes \mathcal{F}$ is δ-Ulrich for $\mathcal{O}(2)$ since $\pi_*\mathcal{E}_m$ is δ-Ulrich for $\mathcal{O}(1)$ and $\mathcal{F}$ is Ulrich for $\mathcal{O}(2)$. Thus $\mathcal{E}_{m+1}$ is δ-Ulrich for $\pi^*\mathcal{O}(2) = \mathcal{O}_X(2^{m+1})$. Moreover,

$$\chi(\mathcal{E}_{m+1}) = \mathrm{rk}(\mathcal{F})(\chi(\mathcal{E}_m) + 3\mathrm{rk}(\pi_*(\mathcal{E}_m))) = \mathrm{rk}(\mathcal{F})(\chi(\mathcal{E}_m) + 3\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))).$$

Since $\deg(\mathcal{O}_X(2^{m+1})) = 4\deg(\mathcal{O}_X(2^m))$, we can write

$$\frac{\chi(\mathcal{E}_{m+1})}{\mathrm{rk}(\mathcal{E}_{m+1})\deg(\mathcal{O}_X(2^{m+1}))} = \frac{1}{4} \cdot \frac{\chi(\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))} + \frac{3}{4}.$$

Now it is clear that

$$\lim_{m\to\infty} \frac{\chi(\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))} = 1.$$

On the other hand we have

$$\frac{\chi(\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_m)\deg(\mathcal{O}_X(2^m))} \leq \alpha(\mathcal{E}_m, \mathcal{O}_X(2^m)) \leq 1$$

and the Theorem follows immediately. □

Remark 4.9. Suppose that $\mathcal{E}$ is a δ-Ulrich sheaf for $\mathcal{O}(1)$ and $\mathcal{F}$ is an $\mathcal{O}(2)$-Ulrich sheaf on $\mathbb{P}^2$. A calculation similar to those in the proof of Lemma 4.7 shows that

$$h^1(\mathcal{E} \otimes \mathcal{F}(-2)) = \mathrm{rk}(\mathcal{F})h^1(\mathcal{E}(-1))$$

Hence

$$\frac{h^1(\mathcal{E} \otimes \mathcal{F}(-2)}{\mathrm{rk}(\mathcal{E} \otimes \mathcal{F})} = \frac{h^1(\mathcal{E}(-1))}{\mathrm{rk}(\mathcal{E})}.$$

So while $\mathcal{E} \otimes \mathcal{F}$ is closer to being $\mathcal{O}(2)$-Ulrich than $\mathcal{E}$ is to being $\mathcal{O}(1)$-Ulrich as measured by $\alpha(-)$, it is no closer at all by this other measure.

Remark 4.10. The minimum rank of an $\mathcal{O}(2)$-Ulrich bundle on $\mathbb{P}^2$ is two. So the ranks of the sheaves $\mathcal{E}_m$ in Theorem 4.8 are growing exponentially.

4.3 Intermediate Cohomology Modules

Let $\mathcal{E}$ be a δ-Ulrich sheaf on X. Our last result describes the structure of the graded module $H^1_*(\mathcal{E})$ in a way that refines Corollary 4.2. First we need a definition.

Definition 4.11. Let S be a standard graded ring and M a finitely generated S module. We say that M has the *Weak Lefschetz Property* [MN13] if there is a linear element $z \in S_1$ such that each multiplication map $\mu_z : M_i \to M_{i+1}$ has maximum rank.

Proposition 4.12. *The graded module* $\mathrm{H}^1_*(\mathcal{E})$ *over the graded ring* $S_X = \mathrm{H}^0_*(\mathcal{O}_X)$ *has the Weak Lefschetz property. Moreover, the following inequalities hold:*

$$h^1(\mathcal{E}(i)) \leq h^1(\mathcal{E}(i+1)) \quad (i \leq -2)$$
$$h^1(\mathcal{E}(i)) \geq h^1(\mathcal{E}(i+1)) \quad (i \geq -2)$$

Proof. Let $H \subset X$ be a hyperplane section (with respect to $\mathcal{O}_X(1)$) such that $\mathcal{E}|_H$ is Ulrich and $z \in \mathrm{H}^0(\mathcal{O}_X(1))$ a defining section. Then consider the long exact sequence

$$\mathrm{H}^0(\mathcal{E}|_H(i+1)) \longrightarrow \mathrm{H}^1(\mathcal{E}(i)) \xrightarrow[\mu_z]{} \mathrm{H}^1(\mathcal{E}(i+1)) \longrightarrow \mathrm{H}^1(\mathcal{E}|_H(i+1)) \qquad (*)$$

on cohomology induced by

$$0 \to \mathcal{E}(i) \to \mathcal{E}(i+1) \to \mathcal{E}|_H(i+1) \to 0.$$

Recall that since $\mathcal{E}|_H$ is Ulrich, we have

$$\mathrm{H}^0(\mathcal{E}|_H(i+1)) = 0,\ (i \leq -2), \quad \text{and} \quad \mathrm{H}^1(\mathcal{E}|_H(i+1)) = 0,\ (i \geq -2).$$

So if $i < -1$, the map μ_z in $(*)$ is injective, and if $i > -3$ it is surjective. □

Remark 4.13. An immediate consequence of Proposition 4.12 is that $\mathrm{H}^1_*(\mathcal{E})$ is generated in degree at most -2. We show this is the best possible statement by exhibiting for each $s \geq 2$ a $\delta-$Ulrich sheaf $\mathcal{E}_s$ such that $\mathrm{H}^1_*(\mathcal{E}_s)$ has a generator in degree $-s$. Consider the simplest of the varieties discussed in Remark 3.6, i.e. a smooth quadric surface $X \subseteq \mathbb{P}^3$. Let L_1, L_2 be the line classes which generate $\mathrm{Pic}(X)$. Then $H = L_1 + L_2$ and $H^\perp$ is generated by $L_1 - L_2$. For each $s \in \mathbb{Z}$, the line bundle $\mathcal{E}_s := \mathcal{O}_X(sL_1 + (1-s)L_2)$ is δ-Ulrich, and fails to be Ulrich precisely when $s \neq 0, 1$. For $s \geq 2$ and $k \in \mathbb{Z}$ we have

$$h^1(\mathcal{E}_s(-s+k)) = h^1(\mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1-2s+k)) = \begin{cases} (k+1)(2s-k-2), & 0 \leq k \leq 2s-3 \\ 0 & \text{otherwise} \end{cases}$$

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